New Entropy Restrictions and the Quest for Better Specified Asset Pricing Models*

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Abstract

In this paper, we feature alternative entropy-based restrictions on the permanent component of stochastic discount factors to evaluate asset pricing models. Specifically, our entropy bound on the square of the permanent component of stochastic discount factors is intended to capture the time-variation in the conditional volatility of the log permanent component as well as distributional non-normalities. Extending extant treatments, we develop entropy codependence measures, our bounds generalize to multi-period permanent component of stochastic discount factors, are based on pricing the risk-free bond, the long-term discount bond, and a set of risky assets, and are substantially sharper. Our empirical application to some state-of-the-art asset pricing models indicates that the search for properly specified asset pricing models is far from over.

KEY WORDS: Entropy, stochastic discount factors, permanent component, lower entropy bounds, entropy codependence, asset pricing models, eigenfunction problem.

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1. Introduction

The quest for well-performing stochastic discount factors (hereby SDFs) has dominated the agenda in asset pricing. For example, Hansen and Jagannathan (1991) promote the result that SDFs must be sufficiently volatile to reconcile the observed equity premium. Alvarez and Jermann (2005) provide an alternative approach, based on the entropy measure, to characterize the permanent component of the SDFs. The central result in Alvarez and Jermann is that the volatility of the permanent component should be large, to rationalize the large spread between the returns of equity and long-term bonds.

Despite substantial progress, identifying the desirable properties of the SDFs and the embedded permanent component, in addition to its link to economic fundamentals, remains an unresolved issue. The search is ongoing, as can be inferred from the treatments in Hansen and Scheinkman (2009), Bakshi and Chabi-Yo (2012), Hansen (2012), and Ghosh, Julliard, and Taylor (2012).

Our approach lies within the tradition of examining the permanent and transitory components of SDFs (e.g., Alvarez and Jermann (2005) and Hansen and Scheinkman (2009)), and we propose new entropy restrictions to evaluate asset pricing models. We offer several results:

- First, our entropy bounds generalize and extend the Alvarez and Jermann (2005) entropy bounds on the permanent component of SDFs to multiple risky assets. In particular, our entropy bounds on the permanent component have no exact analogs and are substantially sharper.
- Second, we develop a new entropy measure based on the square of the permanent component to represent dispersion and to characterize departures from lognormality. We establish that our entropy measure captures time-variation in the conditional volatility of the log permanent component, providing a certain generalization over Alvarez and Jermann (2005).

We further show that our entropy bounds are distinct from the bound derived in Bakshi and Chabi-Yo (2012), who focus on the variance of the permanent component of SDFs. Moreover, we derive two lower entropy bounds, which generalize the entropy bound on the SDF in Backus, Chernov, and Zin (2013). Another characterization extends our bounds approach to a multi-period setting. Furthermore, we develop new entropy codependence measures between the permanent and the transitory components of the SDF.

We illustrate the usefulness of our bounds in the context of three (state-of-the-art) asset pricing models: (i) difference habit, (ii) recursive utility with stochastic variance, and (iii) recursive utility with constant jump intensity (as presented in Backus, Chernov, and Zin (2013)). We solve the eigenfunction problem and derive the permanent and transitory components of the SDF of each model. Our entropy framework provides new perspectives on the performance of these models and their ability to fit asset market quantities.

When the entropy bounds are constructed based on the return properties of the risk-free bond, the long-term discount bond, and multiple risky assets, our implementation reveals that each model produces insufficient entropy to satisfy the lower bound on both (i) the permanent component of the SDF, and (ii) the SDF itself. Our latter finding appears to contradict Backus, Chernov, and Zin (2013), who argue that these models produce too much entropy relative to their single-asset-based entropy bound on the SDF. A block bootstrap-based procedure provides statistical support for our conclusions.

The entropy bound on the square of the permanent component enables a crucial dimension of model assessment. Specifically, the difference habit and the recursive utility with stochastic variance models are rejected. These models can explain only about half of the lower bound estimated from returns data. However, the recursive utility with constant jump intensity model generates entropy that is substantially higher than the lower entropy bound implied from the data. In our search for a possible explanation, we find that this model success can be traced to jump parameterizations of consumption growth that also produce unrealistic distributional higher-moments of the permanent component of the SDF.

One lesson to be drawn is that a viable pricing model must accommodate finite higher-moments of the SDF. We also show that the recursive utility with jump intensity model struggles to match yield curve properties, as gauged by its lack of consistency with the transitory component of the SDF.

Moreover, each model appears to be inconsistent with entropy-based measures of codependence between the permanent and the transitory components of the SDF. These new measures of codependence can be extracted from the returns data, and have no analogs in Hansen (2012).

Our work belongs to a branch of asset pricing that explores the relevance of entropy bounds to discriminate among models. Our value-added is an entropy measure on the square of the permanent component of the SDF (Section 3), and we feature general entropy bounds (Theorem 1). These bounds are aimed at complementing the approaches in Alvarez and Jermann (2005), Bakshi and Chabi-Yo (2012), and Backus, Chernov, and Zin (2013). In the manner of Hansen and Jagannathan (1991, 1997), our formalizations strive to understand model attributes, but our thrust is on the permanent component of the SDF. Moreover, our approach inherits the model-free flavor of Hansen and Jagannathan (1991), our entropy bounds are based on multiple asset returns, we propose a codependence measure (Theorem 2), and we additionally develop a multi-period extension (Theorem 3). The entropy bounds are tractable, can encapsulate data considerations that transcend model calibrations, and our framework can incorporate statistical concerns in model assessment.

2. The entropy of $m^P_{t,t+1}$ in Alvarez and Jermann (2005)

Let $m_{t,t+1}$ represent the stochastic discount factor between date t and $t+1$ and $R_{t,t+1,\infty}$ be the gross return of an infinite-maturity discount bond. Our objective is to propose new entropy measures to evaluate asset pricing models.

As a starting point, we employ a result in Alvarez and Jermann (2005, Proposition 1, page 1983) and Hansen and Scheinkman (2009, page 200), who establish that $m_{t,t+1}$ admits a multiplicative decomposition:

$$m_{t,t+1} = m^P_{t,t+1} m^T_{t,t+1} \qquad \text{with} \qquad E\left[m^P_{t,t+1}\right] = 1 \qquad \text{and} \qquad m^T_{t,t+1} = (R_{t,t+1,\infty})^{-1}, \tag{1}$$

where $m^P_{t,t+1}$ ($m^T_{t,t+1}$) is the permanent (transitory) component of $m_{t,t+1}$ and $E[.]$ is unconditional expectation. The components $m^P_{t,t+1}$ and $m^T_{t,t+1}$ can be correlated, and, if they exist, can be obtained by solving the eigenfunction problem of Hansen and Scheinkman (2009, Corollary 6.1). We center our attention on $m^P_{t,t+1}$, which is a key ingredient for pricing assets in the stock market.

To assess the merits of an asset pricing model, Alvarez and Jermann (2005, page 1985) suggest using the entropy of $m^P_{t,t+1}$, defined as:

$$L[m^P_{t,t+1}] \;=\; \log(E[m^P_{t,t+1}]) \;-\; E[\log(m^P_{t,t+1})] \;=\; -E[\log(m^P_{t,t+1})], \qquad (\text{since } E[m^P_{t,t+1}] = 1)). \tag{2}$$

Alvarez and Jermann show that for some distributions, $L[m^P_{t,t+1}]$ completely characterizes the distribution of $\log(m^P_{t,t+1})$. For example, if $m^P_{t,t+1}$ is lognormally distributed, we must have $\exp(E[\log(m^P_{t,t+1})] + \frac{1}{2}Var[\log(m^P_{t,t+1})]) = 1$. Hence, $L[m^P_{t,t+1}] = -E[\log(m^P_{t,t+1})] = \frac{1}{2}Var[\log(m^P_{t,t+1})]$, and it is only the variance (or equivalently the mean in this setting) of $\log(m^P_{t,t+1})$ that matters for asset pricing.

$L[m^P_{t,t+1}]$ is negatively (positively) related to the third (fourth) moment of $m^P_{t,t+1}$. This trait of the entropy measure can seen by a Taylor expansion of $\log(m^P_{t,t+1})$ around the mean (i.e., $E[m^P_{t,t+1}] = 1$) that is, one may express $L[m^P_{t,t+1}] = -E[\log(m^P_{t,t+1})]$, as:

$$L[m^P_{t,t+1}] = \frac{1}{2}E\left[(m^P_{t,t+1}-1)^2\right] - \frac{1}{3}E\left[(m^P_{t,t+1}-1)^3\right] + \frac{1}{4}E\left[(m^P_{t,t+1}-1)^4\right] + \frac{1}{j}\sum_{j=5}^{\infty}(-1)^j E\left[(m^P_{t,t+1}-1)^j\right], \tag{3}$$

which imparts the observation that $L[m^P_{t,t+1}]$ summarizes the higher-order moments of the $m^P_{t,t+1}$ distribution. To generate higher entropy, modeling approaches often incorporate non-normalities in $m^P_{t,t+1}$ and $m_{t,t+1}$, which could also help to achieve consistency with asset market data (e.g., Wachter (2013)).

3. Motivating the entropy of $(m^P_{t,t+1})^2$ in asset pricing tests

We motivate an alternative entropy-based measure, specifically $L[(m^P_{t,t+1})^2]$, as a metric for evaluating asset pricing models, in conjunction with $L[m^P_{t,t+1}]$. In analogy to equation (2), we define

$$L[(m^P_{t,t+1})^2] \;=\; \log(E[(m^P_{t,t+1})^2]) \;-\; E[\log((m^P_{t,t+1})^2)]. \tag{4}$$

While developing the implications of this new entropy measure, our analysis centers around two key issues. First, what do we miss when the entropy measure $L[m^P_{t,t+1}]$ is employed to assess the consistency of $m^P_{t,t+1}$ of an asset pricing model with observed asset prices? Second, what do we gain when $L[(m^P_{t,t+1})^2]$ is applied to asset pricing problems? Our framework is also pertinent to understanding how one could use observed asset prices to learn about dependence in $m^P_{t,t+n}$ (or $m_{t,t+n}$) over any generic investment horizon n.

The next example first showcases an environment where $L[(m^P_{t,t+1})^2]$ has no role beyond $L[m^P_{t,t+1}]$. Steps leading to most of the results that follow are shown in Online Appendix I.

Example 1 Let the dynamics of the permanent and transitory components be given by (Alvarez and Jermann (2001, page 9); see also Campbell (1986, equation (3))):

$$\log\left(m^P_{t,t+1}\right) = -\frac{1}{2}\sigma_P^2 + \varepsilon^P_{t+1} \quad \text{and} \quad \log\left(m^T_{t,t+1}\right) = \log(\beta) + \alpha_0\,\varepsilon^T_{t+1} + \sum_{i=1}^{\infty}(\alpha_i - \alpha_{i-1})\,\varepsilon^T_{t+1-i}, \tag{5}$$

where the two shocks are normally distributed and homoskedastic, i.e., $\varepsilon^P_{t+1} \sim \mathcal{N}(0,\sigma_P^2)$ and $\varepsilon^T_{t+1} \sim \mathcal{N}(0,\sigma_T^2)$, with constant correlation. Then

$$L[(m^P_{t,t+1})^2] \;=\; 4\,L[m^P_{t,t+1}] \qquad \text{where} \qquad L[m^P_{t,t+1}] \;=\; \frac{\sigma_P^2}{2}. \tag{6}$$

Equation (6) shows that when the conditional volatility of $\log(m^P_{t,t+1})$ is time-invariant, the two entropy measures $L[m^P_{t,t+1}]$ and $L[(m^P_{t,t+1})^2]$ contain identical information, i.e., $L[(m^P_{t,t+1})^2] - 4\,L[m^P_{t,t+1}] = 0$. ♣

To address possible advantages of $L[(m^P_{t,t+1})^2]$ over $L[m^P_{t,t+1}]$ from the vantage point of asset pricing, we apply the definition of $L[u]$ to random variables u^2 and u, and arrive at the following result:

$$L[(m^P_{t,t+1})^2] \;-\; 4L\left[m^P_{t,t+1}\right] \;=\; \log(E\left[(m^P_{t,t+1})^2\right]) \;+\; E[\log((m^P_{t,t+1})^2)]. \tag{7}$$

Equation (7) indicates that the departure between $L[(m^P_{t,t+1})^2]$ and $4L[m^P_{t,t+1}]$ can be attributed to the time-variation in the conditional volatility of $\log(m^P_{t,t+1})$. The following example puts this notion on a solid footing.

Example 2 Suppose an eigenfunction problem yields $\log(m^P_{t,t+1}) \sim \mathcal{N}(\mu_t, \sigma_t^2)$. Then

$$L[(m^P_{t,t+1})^2] = \log\left(E\left[e^{\sigma_t^2}\right]\right) \;+\; E[\sigma_t^2] \qquad \text{and} \qquad L[m^P_{t,t+1}] = \frac{1}{2}E[\sigma_t^2]. \tag{8}$$

Using Taylor expansion of $e^{\sigma_t^2}$ around $\sigma_t^2 = E\left[\sigma_t^2\right]$, we observe

$$L[(m^P_{t,t+1})^2] \;-\; 4L[m^P_{t,t+1}] \;=\; \log\left(1 + \sum_{j=2}^{\infty} \frac{1}{j!} E\left[(\sigma_t^2 - E[\sigma_t^2])^j\right]\right). \tag{9}$$

The information embedded in the distribution of σ_t^2 differentiates $L[(m^P_{t,t+1})^2]$ from $L[m^P_{t,t+1}]$. In general, $L[m^P_{t,t+1}]$ and $L[(m^P_{t,t+1})^2]$ contain distinct information relevant to distinguishing asset pricing models. ♣

We further note that $L[m^P_{t,t+1}]$ and $L[(m^P_{t,t+1})^2]$ do not coincide because $L[m^P_{t,t+1}] > 0$, hence, $L[m^P_{t,t+1}]$ and $L[(m^P_{t,t+1})^2]$ reflect distinct entropies. More generally, $L[(m^P_{t,t+1})^2]$ subsumes $L[m^P_{t,t+1}]$.

The entropy measure $L[(m^P_{t,t+1})^2]$ offers flexibility in detecting non-normalities in $\log(m^P_{t,t+1})$. From a Taylor expansion of $\exp\left(\log((m^P_{t,t+1})^2)\right)$ around $E[\log(m^P_{t,t+1})]$, we note that equation (4) implies:

$$L[(m^P_{t,t+1})^2] = \log\left(1 + \sum_{j=1}^{\infty} \frac{2^j}{j!}\kappa_j\right), \qquad \text{where} \qquad \kappa_j \equiv E[(\log(m^P_{t,t+1}) - E[\log(m^P_{t,t+1})])^j] \tag{10}$$

is the jth central moment of $\log(m^P_{t,t+1})$. The normality of $\log(m^P_{t,t+1})$ imposes two restrictions, first, that $\kappa_j = 0$ for $j \geq 3$, and, second, that $L[m^P_{t,t+1}] = -E[\log(m^P_{t,t+1})] = \frac{1}{2}Var[\log(m^P_{t,t+1})]$. Therefore, under the normality of $\log(m^P_{t,t+1})$,

$$L[(m^P_{t,t+1})^2] \;=\; \log\left(1 \;+\; 2Var\left[\log\left(m^P_{t,t+1}\right)\right]\right) \;=\; \log\left(1 - 4E\left[\log\left(m^P_{t,t+1}\right)\right]\right) \;\approx\; 4L[m^P_{t,t+1}]. \tag{11}$$

Thus, $L[(m^P_{t,t+1})^2]$ may be construed as capturing the departure of $\log(m^P_{t,t+1})$ from normality.

A further argument could be made that a measure suitable for evaluating the consistency of permanent component of SDFs must be positively related to skewness. In particular, using $L[m^P_{t,t+1}]$ and $L[(m^P_{t,t+1})^2]$ together allows to account for the skewness and fat-tails of $m^P_{t,t+1}$ that may not be adequately characterized by $L[m^P_{t,t+1}]$ (see our equation (3)). This can be particularly relevant when evaluating asset pricing models with stochastic volatility and jumps in the dynamics of consumption growth (and possibly other forcing state variables).

There is also an exact relation between $L[m^P_{t,t+1}]$, $L[(m^P_{t,t+1})^2]$ and $Var[m^P_{t,t+1}]$, illustrating that asset pricing models might satisfy restrictions on $L[m^P_{t,t+1}]$ and not on $L[(m^P_{t,t+1})^2]$. It may be verified that

$$L[(m^P_{t,t+1})^2] \; - \; 2L[m^P_{t,t+1}] \; = \; \log\left(1 \; + \; Var\left[m^P_{t,t+1}\right]\right) \; \approx \; Var\left[m^P_{t,t+1}\right]. \tag{12}$$

When $m^P_{t,t+1}$ is the permanent component with the lowest variance, $Var[m^P_{t,t+1}]$ corresponds to the *minimum* variance of $m^P_{t,t+1}$ in Bakshi and Chabi-Yo (2012, equation (6)).

The following example further synthesizes the various elements of our analysis.

Example 3 Suppose the SDF is governed by (Backus, Foresi, and Telmer (2001, equation (19))),

$$\log(m_{t,t+1}) \; = \; -\delta \; - \; \gamma z_t \; - \; \lambda z_t^{\frac{1}{2}}\, \varepsilon_{t+1}, \quad z_{t+1} \; = \; (1-\varphi)\,\theta \; + \; \varphi z_t \; + \; \sigma z_t^{\frac{1}{2}}\, \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim \mathcal{N}(0,1), \tag{13}$$

for a state variable z_t. We solve the eigenfunction problem to derive (see the Online Appendix II):

$$m^T_{t,t+1} = \exp\left(-\delta + \xi\,(1-\varphi)\,\theta + \xi\,(z_t - z_{t+1})\right), \qquad m^P_{t,t+1} = \exp\left((-\xi - \gamma + \xi\varphi) z_t + (\xi\sigma - \lambda)\, z_t^{\frac{1}{2}}\, \varepsilon_{t+1}\right), \tag{14}$$

where $\xi \equiv \frac{-(\varphi - 1 - \lambda\sigma) - \sqrt{(\varphi - 1 - \lambda\sigma)^2 - 2\sigma^2\left(\frac{1}{2}\lambda^2 - \gamma\right)}}{\sigma^2}$ and $\gamma \equiv \frac{1}{2}\left(1 + \lambda^2\right)$. It can be further shown that

$$L[(m^P_{t,t+1})^2] \; - \; 2L[m^P_{t,t+1}] \; = \; \log\left(E\left[e^{\left(2(-\xi-\gamma+\xi\varphi) \; + \; 2\,(\xi\sigma-\lambda)^2\right) z_t}\right]\right). \tag{15}$$

The stochastic nature of z_t can introduce a wedge between $L[(m^P_{t,t+1})^2]$ and $L[m^P_{t,t+1}]$. ♣

In summary, our analytical links highlight that if one is interested in using entropy to learn about properties of the SDFs, one may need to use $L[(m^P_{t,t+1})^2]$ in conjunction with $L[m^P_{t,t+1}]$. The essential distinguishing trait of the entropy measure $L[(m^P_{t,t+1})^2]$ is its ability to more effectively cope with the effect of time-varying volatility and distributional non-normalities in $\log(m^P_{t,t+1})$.[1]

4. Entropy bounds for a generic set of returns

Our object of interest is a bound on $L[(m^P_{t,t+1})^2]$ and $L[m^2_{t,t+1}]$, and we illustrate the advantages of these entropy bounds compared to the corresponding bounds on $L[m^P_{t,t+1}]$ and $L[m_{t,t+1}]$. We also develop restrictions that are based on the entropy codependence between $m^P_{t,t+1}$ and $m^T_{t,t+1}$.

To proceed with the development of our entropy bounds, consider a set $\mathbb{S}$ of SDFs that prices correctly returns:

$$\mathbb{S} = \{m_{t,t+1} > 0 : E_t\left[m_{t,t+1}\right] = q_t,\ \ E\left[m_{t,t+1}R_{t,t+1,\infty}\right] = 1,\ \text{ and }\ E\left[m_{t,t+1}\mathbf{R}_{t,t+1}\right] = \mathbf{1}\}\,, \tag{16}$$

where $\mathbf{1}$ is a vector column of ones. Moreover, $\mathbf{R}_{t,t+1}$ is an $N \times 1$ vector of gross returns that excludes the risk-free bond and the infinite-maturity discount bond. We further postulate that some SDFs that belong to $\mathbb{S}$ can be decomposed into permanent and transitory components:

$$\mathbb{S}_P = \left\{m_{t,t+1} \in \mathbb{S} : m_{t,t+1} = m^P_{t,t+1}\, m^T_{t,t+1},\ \text{ and }\ m^T_{t,t+1} = (R_{t,t+1,\infty})^{-1}\right\}. \tag{17}$$

The formulations in equations (16) and (17) allow us to develop entropy bounds that are based on the return properties of $N+2$ assets and, hence, offer considerable generality, and may be sharper.

The motivation for considering $\mathbb{S}_P$ stems from Alvarez and Jermann (2005) and Bakshi and Chabi-Yo (2012), who show that the bounds on $m^P_{t,t+1}$ can be useful for understanding the behavior of asset prices. The treatment of Hansen (2012) further highlights the relevance of permanent and transitory components of $m_{t,t+1}$, potentially helping to narrow the search for properly specified asset pricing models.

For the characterizations to follow, define

$$\mathbf{y} \equiv \Sigma^{-1}\left(\mathbf{1} - E\left[q_t\right] E\left[\mathbf{R}_{t,t+1}\right]\right), \qquad \mathbf{y}_P \equiv \Sigma_P^{-1}\left(\mathbf{1} - E\left[\mathbf{R}_{t,t+1}/R_{t,t+1,\infty}\right]\right), \qquad \text{and} \qquad \mathbf{w} \equiv \frac{\mathbf{y}}{\mathbf{1}'\mathbf{y}}, \tag{18}$$

where Σ is the variance-covariance matrix of $\mathbf{R}_{t,t+1}$, and Σ_P is the variance-covariance matrix of $\mathbf{R}_{t,t+1}/R_{t,t+1,\infty}$.

[1] Analogously, $L[m^2_{t,t+1}]$ captures asymmetries in $\log(m_{t,t+1})$. We note that $L[m^2_{t,t+1}] = \log(1 + \sum_{j=1}^{\infty} \frac{2^j}{j!} E(\log(m_{t,t+1}) - E[\log(m_{t,t+1})])^j)$, and therefore $L[m^2_{t,t+1}] = 4L[m_{t,t+1}]$ under the normality of $\log(m_{t,t+1})$. In Colacito, Ghysels, and Meng (2013, equation (11)), the $\log\left(m_{t,t+1}\right)$ is not normal and $L[m^2_{t,t+1}] \neq 4L[m_{t,t+1}]$ illustrating that $L[m^2_{t,t+1}]$ could be suitable candidate for evaluating asset pricing models under deviations from lognormality.

We assume that $\mathbf{w}'\mathbf{R}_{t,t+1}$ is strictly positive. Define

$$\mathrm{er}_R \equiv E\left[\log\left(\mathbf{w}'\mathbf{R}_{t,t+1}\right)\right] \;-\; \log\left((E[q_t])^{-1}\right) \qquad \text{and} \qquad \mathrm{v}_R \equiv \log\left(1+\mathbf{y}'\Sigma\mathbf{y}/(E[q_t])^2\right), \tag{19}$$

$$\mathrm{er}_\infty \equiv E\left[\log\left(R_{t,t+1,\infty}\right)\right] \;-\; \log\left((E[q_t])^{-1}\right) \qquad \text{and} \qquad \mathrm{v}_P \equiv \log\left(1+\mathbf{y}_P'\Sigma_P\mathbf{y}_P\right). \tag{20}$$

Now we derive the bounds on the entropies $L[(m^P_{t,t+1})^2]$ and $L[m^2_{t,t+1}]$.

Theorem 1 *The following bounds are germane to asset pricing models:*

(a) The entropy of $(m^P_{t,t+1})^2$ and $m^P_{t,t+1}$ satisfy:

$$L[(m^P_{t,t+1})^2] \;\geq\; 2\left(\mathrm{er}_R \;-\; \mathrm{er}_\infty\right) \;+\; \mathrm{v}_P \qquad \text{and} \qquad L[m^P_{t,t+1}] \;\geq\; \mathrm{er}_R \;-\; \mathrm{er}_\infty. \tag{21}$$

(b) The entropy of $m^2_{t,t+1}$ and $m_{t,t+1}$ satisfy:

$$L[m^2_{t,t+1}] \;\geq\; 2\,\mathrm{er}_R \;+\; \mathrm{v}_R \qquad \text{and} \qquad L[m_{t,t+1}] \;\geq\; \mathrm{er}_R, \tag{22}$$

where er_R *and* v_R *are defined in equation (19) and* er_∞ *and* v_P *are defined in equation (20).*

Proof: See Appendix A. ■

The entropy bounds stipulated in equations (21) and (22) summarize properties of the distribution of $m^P_{t,t+1}$ and $m_{t,t+1}$ and, hence, contain information that could help to gauge asset pricing models. Moreover, the lower bounds presented in the right-hand side of equations (21) and (22) in Theorem 1 are computable from the time-series of asset returns and discount bonds. In addition, our bounds are model-free.

Equation (21) features the bound on $L[(m^P_{t,t+1})^2]$, which can be employed in conjunction with the bound on $L[m^P_{t,t+1}]$ to evaluate the consistency of permanent component of an SDF with observed asset prices. Likewise, the bound on $L[m^2_{t,t+1}]$ can be employed in conjunction with the bound on $L[m_{t,t+1}]$ to evaluate whether SDF properties are consistent with observed asset prices. Our bounds rely on the ability of the SDF to correctly price the risk-free bond, the long-term discount bond, and a set of risky asset returns.

Theorem 1 offers the distinction that the entropy $L[(m^P_{t,t+1})^2]$ is bounded by the mean and a quadratic form of mean and variance of asset returns, whereas $L[m^P_{t,t+1}]$ is bounded by the mean of asset returns.

One may interpret the lower bound on $L[(m^P_{t,t+1})^2]$ as having two economically meaningful components. The first term represents the difference between the excess log return on a risky portfolio and the excess log return of an infinite-maturity discount bond, whereas the second term is proportional to a Sharpe ratio-related component.

Our bounds on $L[m^P_{t,t+1}]$ and $L[m_{t,t+1}]$ could be viewed as a generalization of the single risky asset counterparts in Alvarez and Jermann (2005) on $L[m^P_{t,t+1}]$, and in Backus, Chernov, and Zin (2013) on $L[m_{t,t+1}]$.

- When $\mathbf{R}_{t,t+1}$ specializes to a single risky asset, our lower entropy bound on $L[m^P_{t,t+1}]$ specializes to the one in Alvarez and Jermann (2005, equation (4)):

$$\text{Alvarez and Jermann: } \text{er}_R \;-\; \text{er}_\infty \text{ in eq. (21) becomes } E[\log\left(R^m_{t,t+1}\right)] \;-\; E[\log\left(R_{t,t+1,\infty}\right)], \quad (23)$$

  where $R^m_{t,t+1}$ is the gross return of the stock market. We will show that our bound on $L[m^P_{t,t+1}]$ that relies on the return properties of $N+2$ assets is considerably more stringent.

- Two differences emerge with respect to the entropy bound on $m_{t,t+1}$ in Backus, Chernov, and Zin (2013). First, when there is a single risky asset, notably the stock market, our bound on $L[m_{t,t+1}]$ is:

$$\text{er}_R \;=\; E[\log\left(R^m_{t,t+1}\right)] \;-\; \log\left(1/E\left[q_t\right]\right) \quad \text{versus} \quad \text{er}^{BCZ}_R \;=\; E[\log\left(R^m_{t,t+1}\right)] \;-\; E[\log(R^f_t)] \quad (24)$$

  in Backus, Chernov, and Zin (2013, equation (5)). Since $R^f_t = 1/q_t$, Jensen's inequality implies $E[\log\left(q_t\right)] \leq \log\left(E\left[q_t\right]\right)$ and, hence, $\text{er}_R \geq \text{er}^{BCZ}_R$. Second, our bound on $L[m_{t,t+1}]$ in equation (22) extends the Backus, Chernov, and Zin (2013, equation (5)) bound to many risky assets.

Apart from theoretical arguments, how sharp are our generalized bounds compared to the ones in Alvarez and Jermann (2005) and Backus, Chernov, and Zin (2013)? To address this question, we report our lower bounds on $L[m^P_{t,t+1}]$ corresponding to three datasets, and the associated bootstrap p-values, in Table 1. The lesson to be drawn from our computations is that our bounds on $L[m^P_{t,t+1}]$ and $L[m_{t,t+1}]$ are intrinsically sharper, implying greater hurdles on asset pricing models. We will show that considering $L[(m^P_{t,t+1})^2]$ and $L[(m_{t,t+1})^2]$ can further help to discern across asset pricing models.

Recognize also that the lower bound on $L[(m^P_{t,t+1})^2]$ in equation (21) is distinct from the lower bound on $Var[m^P_{t,t+1}]$ in Bakshi and Chabi-Yo (2012, equation (6)). Analogously, the lower bound on $Var[m_{t,t+1}]$, that is, the Hansen and Jagannathan (1991, equation (12)) bound, and our lower bound on $L[(m_{t,t+1})^2]$ in equation (22), constitute distinctly relevant metrics for evaluating asset pricing models.

The analysis in Backus, Chernov, and Zin (2013, Section A.2) points to the distinct nature of the lower bound on $Var[m_{t,t+1}]$ versus the lower bound on $L[m_{t,t+1}]$ (see also a discussion in Alvarez and Jermann (2005, page 1985)). As noted in equation (12), the entropy of $m^P_{t,t+1}$, the entropy of $(m^P_{t,t+1})^2$, and the variance of $m^P_{t,t+1}$ are related by the expression: $\exp(L[(m^P_{t,t+1})^2]-2L[m^P_{t,t+1}])-1=Var[m^P_{t,t+1}]$. Such a relation implies that it may be possible for a model to satisfy the bound on $Var[m^P_{t,t+1}]$, but not the bound on $L[(m^P_{t,t+1})^2]$ and vice versa.

Our analysis can be adapted to develop restrictions on the entropy of $(m^T_{t,t+1})^2$ and $m^T_{t,t+1}$. Intuitively, both $L[m^T_{t,t+1}]$ and $L[(m^T_{t,t+1})^2]$ capture the departure of $R_{t,t+1,\infty}$ from lognormality. Moreover, when there is no time-variation in the conditional variance of $\log(R_{t,t+1,\infty})$, we obtain equivalence of the type $L[(m^T_{t,t+1})^2]=4L[m^T_{t,t+1}]$. Absent distributional assumptions, the general restrictions are

$$L[(m^T_{t,t+1})^2] = \log(E\left[1/R^2_{t,t+1,\infty}\right]) + 2E[\log(R_{t,t+1,\infty})], \quad \text{and} \tag{25}$$

$$L[m^T_{t,t+1}] = \log(E\left[1/R_{t,t+1,\infty}\right]) + E[\log(R_{t,t+1,\infty})]. \tag{26}$$

The restrictions (25) and (26) inherit the model-free attribute of the entropy bounds on $(m^P_{t,t+1})^2$ and $m^P_{t,t+1}$. Given a proxy for $R_{t,t+1,\infty}$, the quantities on the right-hand side of (25)–(26) are computable. These quantitative restrictions can be helpful in investigating whether a pricing model is aligned with the properties of the transitory component of the SDFs, as reflected in the return time-series of long-term discount bonds.

Inspired by a treatment in Hansen (2012, Section 4.3), we develop two additional results in the context of the permanent and transitory components of the SDF. First, we note that

$$\underbrace{L\left[m^P_{t,t+1}m^T_{t,t+1}\right] - L\left[m^P_{t,t+1}\right] - L\left[m^T_{t,t+1}\right]}_{\text{Intrinsic to a model}} = \underbrace{\log(E\left[q_t\right]) - \log(E\left[1/R_{t,t+1,\infty}\right])}_{\text{Can be recovered from bond data}}, \tag{27}$$

where recognizing that the left-hand side of equation (27) simplifies to $\log(E[m^P_{t,t+1}m^T_{t,t+1}])-\log(E[m^P_{t,t+1}])-\log(E[m^T_{t,t+1}])$ by virtue of the definition of entropy, while the right-hand side of equation (27) can be inferred from the term structure of default-free interest rates. Second, we develop an upper bound on the entropy-based codependence between $(m^P_{t,t+1})^2$ and $(m^T_{t,t+1})^2$ and state it as a formal result.

Theorem 2 *The following upper bound on the entropy-based codependence measure is true:*

$$0 \leq L[(m^P_{t,t+1})^2(m^T_{t,t+1})^2]-L\left[(m^P_{t,t+1})^2\right]-L[(m^T_{t,t+1})^2] \leq \log\left(1+\frac{\mathbf{y}'\Sigma\mathbf{y}}{(E\left[q_t\right])^2}\right), \tag{28}$$

where $\mathbf{y}$ *is defined in equation (18) and* Σ *is the variance-covariance matrix of* $\mathbf{R}_{t,t+1}$.

Proof: See Appendix B. ■

The two codependence measures capture fundamentally different information embedded in an asset pricing model. Specifically, the restriction in equation (27) traces codependence exclusively to bond prices, while inequality in equation (28) of Theorem 2 traces codependence predominantly to the mean and variance-covariance matrix of a generic set of risky asset returns.

5. Permanent component of asset pricing models

Our goal is to learn about the properties of $m^P_{t,t+1}$ and $m^T_{t,t+1}$, and their consistency with the entropy restrictions and entropy codependence measures. We focus on three asset pricing models: (i) difference habit, (ii) recursive utility with stochastic variance, and (iii) recursive utility with constant jump intensity. Our analysis can be expanded to consider other asset pricing models. We complement the analysis in Backus, Chernov, and Zin (2013) by solving the eigenfunction problem and by studying the implications of entropy codependence measures (for ease of exposition, we also closely follow their model notation).

5.1. Difference habit model

In the difference habit model (e.g., Campbell and Cochrane (1999)), the SDF is

$$m_{t,t+1} \;=\; \beta\, g_{t+1}^{\rho-1}\,(s_{t+1}/s_t)^{\rho-1}\,, \tag{29}$$

where g_{t+1} is consumption growth, β is the time discount parameter, and $1-\rho$ is the coefficient of relative risk aversion. Define $s_t \equiv 1-\exp(z_t)$ and $z_t \equiv \log(h_t)-\log(c_t)$, where s_t is the surplus ratio corresponding to z_t, and the habit h_{t+1} is known at time t. The laws of motion for h_t and g_t are

$$\log(h_{t+1}) = \log(h)+\eta\,[B]\,\log(c_t) \qquad \text{and} \qquad \log(g_{t+1}) = \log(g)+\gamma[B]\,\upsilon^{\frac{1}{2}}\omega_{gt+1}, \tag{30}$$

where B is the lag operator, such that $B\{s_{t+1}\} = s_t$, with backshift operators $\gamma[B] = \sum_{j=0}^{\infty}\gamma_j B^j$ and $\eta\,[B] = \sum_{j=0}^{\infty}\eta_j B^j$. Moreover, υ denotes the constant variance of $\log(g_t)$, and ω_{gt+1} is i.i.d. standard normal variable.

Loglinear approximation of $\log(s_t)$, in conjunction with the laws of motion in equation (30), leads to the surplus ratio dynamics:

$$\log(s_{t+1}) - \log(s_t) = \left(\frac{s-1}{s}\right)(\eta\,[B]\,B-1)\log(g_{t+1}). \tag{31}$$

Completing model description, we define the state variable $x_t = (\gamma[B]-\gamma_0)\,\upsilon^{\frac{1}{2}}\,\omega_{gt+1}$, which governs the dynamics of the log consumption growth:

$$x_t = \gamma_1\,\upsilon^{\frac{1}{2}}\,\omega_{gt} + \varphi_g\,x_{t-1} \qquad \text{with} \qquad \varphi_g = \frac{\gamma_2}{\gamma_1}. \tag{32}$$

Solving the eigenfunction problem (as formalized in equations (D1) and (D2)) results in the following:

Proposition 1 *For the SDF of the habit model specified in equation (29), the permanent component is:*

$$m^P_{t,t+1} \quad = \quad \exp\left(-D_1 + D_2\,x_{t-1} + D_3\,x_t + D_4\,x_{t+1}\right), \tag{33}$$

where the dynamics of x_t is displayed in equation (32) and the coefficients D_1 through D_4 are defined in equations (E15) through (E18) of Online Appendix III.

Proof: See the steps in Online Appendix III. ■

We employ equation (33) of Proposition 1 to compute the left-hand side of the bound expressions (21)-(22) of Theorem 1. Asset pricing models that accommodate habit have shown promise in matching salient attributes of the asset market data, including the equity premium, procyclicality of stock prices, countercyclicality of stock volatility, and return predictability at long-horizons (e.g., see, among others, Bekaert and Engstrom (2012), Chapman (1998), Chan and Kogan (2002), and Santos and Veronesi (2010)).

5.2. Recursive utility models

The two recursive utility models that we consider are adopted from Backus, Chernov, and Zin (2013):

$$U_t = \left[(1-\beta)\,c_t^{\rho} \;+\; \beta\,(\mu_t\,[U_{t+1}])^{\rho}\right]^{\frac{1}{\rho}}, \tag{34}$$

with certainty equivalent function $\mu_t\,[U_{t+1}] = \left(E_t\left[U^{\alpha}_{t+1}\right]\right)^{\frac{1}{\alpha}}$. Moreover, ρ is the time preference parameter, $1/(\rho-1)$ is the intertemporal elasticity of substitution, and $1-\alpha$ is the coefficient of relative risk aversion.

With backshift operators characterized by $\nu\,[B] = \sum_{j=0}^{\infty} \nu_j B^j$ and $\psi\,[B] = \sum_{j=0}^{\infty} \psi_j B^j$, the state-variables in this model obey the dynamics:

$$\log(g_t) = \log(g) + \gamma[B]\,\upsilon_{t-1}^{1/2}\omega_{gt} + \psi[B]\,z_{gt} - \psi[1]\,h\theta, \qquad h_t = h + \eta\,[B]\,\omega_{ht}, \tag{35}$$

$$\upsilon_t = \upsilon \,+\, \nu\,[B]\,\omega_{\upsilon t}, \quad z_{gt}|j \sim \mathcal{N}\left(j\theta, j\delta^2\right), \quad P\,[j] = \exp\left(-h_{t-1}\right)\frac{(h_{t-1})^j}{j!}, \tag{36}$$

where ω_{gt}, z_{gt}, and ω_{ht} are standard normal random variables, independent of each other and across time. The jump component z_{gt} is a Poisson mixture of normals: conditional on the number of jumps j, z_{gt} is normal, with mean $j\theta$ and variance $j\delta^2$. The probability of $j \geq 0$ jumps at date t is $e^{h_{t-1}}h_{t-1}^j/j!$, and the jump intensity, h_{t-1}, is the mean of j.

A. Recursive utility model with stochastic variance. Set $h = 0$, $\eta\,[B] = 0$, $\psi\,[B] = 0$ in equations (35) and (36). For tractability, we consider the evolution of the transformed variable:

$$x_t \;=\; \varphi_g\,x_{t-1} \;+\; \gamma_1\,\upsilon_{t-1}^{1/2}\,\omega_{gt}. \tag{37}$$

Now we prove.

Proposition 2 *For the SDF of the recursive utility model with stochastic variance, the permanent component is:*

$$m_{t,t+1}^P \quad = \quad \exp\left(H_6 + (H_2 - \tau_0)\,x_t \;+\; (H_3 + \tau_0)\,x_{t+1} + (H_4 - \tau_1)\,\upsilon_t + (H_5 + \tau_1)\,\upsilon_{t+1}\right), \tag{38}$$

where the coefficients H_2 through H_6, τ_0, and τ_1 are presented in Online Appendix IV.

Proof: See the steps in the Online Appendix IV. ■

B. Recursive utility model with constant jump intensity: In equations (35) and (36), set $\nu\,[B] = 0$. We obtain the following result.

Proposition 3 *For the SDF of the recursive utility model with constant jump intensity, the permanent component is:*

$$m_{t,t+1}^P \quad = \quad \exp\Big(G_9 - G_8\,h_t + (G_5 + \varsigma_1)\,z_{gt+1} + (G_6 + \varsigma_2\gamma_1)\,\upsilon^{\frac{1}{2}}\,\omega_{gt+1} + (G_7 + \varsigma_0\eta_0)\,\omega_{ht+1}\Big), \tag{39}$$

where the coefficients G_5 through G_9, ς_0 through ς_3, and η_0 are presented in Online Appendix IV.

Proof: See the steps in Online Appendix IV. ■

Asset pricing models that incorporate recursive preferences in conjunction with stochastic variance or jumps in the consumption growth dynamics have proved successful in explaining asset pricing quantities. Notable applications include, among others, Epstein and Zin (1991), Bansal and Yaron (2004), Campbell and Vuolteenaho (2004), Hansen, Heaton, and Li (2008), Wachter (2013), Zhou and Zhu (2009), and references contained therein. In particular, Wachter (2013) emphasizes that her model can reconcile the size of the equity premium, the behavior of equity volatility, and the return predictability of Treasury bonds, pointing to a possible link between seemingly disparate phenomena from equity and bond markets.

6. Analyzing asset pricing models

Our benchmark for assessing whether a model produces too much entropy are the bounds in Theorem 1 that are computed based on $N+2$ asset returns. Moreover, we consider a block bootstrap procedure to judge whether a model statistically meets our lower entropy bounds. We then juxtapose our analysis with new entropy-based measures of codependence, which are motivated by a discussion in Hansen (2012).

Pertinent to our empirical inquiry is first the question: How meaningful are our entropy bound on $(m^P_{t,t+1})^2$? To address this question, we need to show that the entropy $L[(m^P_{t,t+1})^2]$ contains information beyond that which is contained in the entropy $L[(m^P_{t,t+1})]$. Note that in a setting where $m^P_{t,t+1}$ is lognormally distributed with no time-variation in the conditional volatility of $\log(m^P_{t,t+1})$, one obtains the restriction: $L[(m^P_{t,t+1})^2] - 4L[m^P_{t,t+1}] = 0$ (see Example 1). One implication of this restriction is that the lower bound on $L[(m^P_{t,t+1})^2]$ is proportional to the lower bound on $L[m^P_{t,t+1}]$, which is amenable to validation from the returns data. Guided by this reasoning, and combining the right-hand sides of equation (21) in Theorem 1, we consider the quantity

$$\Pi \equiv \frac{2\,(\mathrm{er}_R - \mathrm{er}_\infty) + \mathrm{v}_P}{4(\mathrm{er}_R - \mathrm{er}_\infty)} - 1. \tag{40}$$

The hypothesis $\Pi = 0$ amounts to testing whether $L[(m^P_{t,t+1})^2]$ and $L[m^P_{t,t+1}]$ impound the same information. Table 2 provides a point estimate of Π for three sets of $\mathbf{R}_{t,t+1}$ and a bootstrap p-value that tests whether $\Pi = 0$ versus $\Pi \neq 0$. Our empirical analysis elicits the observation that the hypothesis of $\Pi = 0$ is rejected, whereby $L[(m^P_{t,t+1})^2]$ can depart from $4L[m^P_{t,t+1}]$ by as much as 51.85%. The reported p-values are based on a block bootstrap, with a block size of 20, with 50,000 replications from the data. Our evidence provides some rationale for considering the entropy $L[(m^P_{t,t+1})^2]$ in assessing asset pricing models.

How do the models under consideration fare when viewed from the perspective of our entropy bounds? Our implementation of the models with difference habit (hereby, DH), recursive utility with stochastic variance (hereby, RU-SV), and recursive utility with constant jump intensity (hereby, RU-CJI) follows the calibration procedure in Backus, Chernov, and Zin (2013, respectively, Model (4) in Table 2, Model (1) in Table 3, and Model (4) in Table 4). The corresponding model parameterizations are displayed in our Table Appendix-I, which indicates that each model reasonably calibrates to consumption growth data.

Aided by the analytical representations of $m^P_{t,t+1}$ derived in our Propositions 1 through 3, we generate the paths for $m^P_{t,t+1}$ based on the model parameters in Table Appendix-I. Then we compute the four entropies $L[(m^P_{t,t+1})^2]$, $L[m^P_{t,t+1}]$, $L[(m_{t,t+1})^2]$, and $L[m_{t,t+1}]$ (for example, according to equation (4)). We draw 50,000 paths for the shocks driving a model (e.g., $\omega_{\upsilon t}$ and ω_{gt} for the RU-SV) and, hence, obtain 50,000 paths for $m^P_{t,t+1}$ and $m_{t,t+1}$. The p-values, shown in square brackets, represent the proportion of replications for which the model-based entropy measure exceeds the corresponding lower bound estimated from the returns data in 50,000 replications of a simulation over 966 months (i.e., over 1931:07 to 2011:12).

The next question to ask is: Do the three models generate too much entropy $L[m_{t,t+1}]$? Panel B of Table 3 reveals that the average $L[m_{t,t+1}]$ is 0.0196, 0.0217, and 0.0190, respectively, for the DH, RU-SV, and RU-CJI models. Based on our lower bound on $L[m_{t,t+1}]$ in equation (22), computed based on SET B, all the models are rejected at the 5% level (as seen by the bootstrap p-values). Thus, the implications from our generalized bounds (based on the return properties of the risk-free bond, the long-term discount bond, the equity market, and the 25 portfolios sorted by size and momentum) diverge from a conclusion in Backus, Chernov, and Zin (2013) that these asset pricing models generate too much entropy.

How does one explain this discrepancy? We note that the magnitude of the lower bound on $L[m_{t,t+1}]$ in the calculations of Backus, Chernov, and Zin (2013, Table 1, row S&P 500) is 0.0040, whereas it is 0.0367, based on our lower bound and SET B. It bears emphasizing that a single-asset based lower bound on $L[m_{t,t+1}]$ provides an insufficient hurdle in evaluating the merits of an asset pricing model.

We are now prompted to ask: Are the properties of $m^P_{t,t+1}$ implicit in the models consistent with the entropy bound $L[m^P_{t,t+1}]$? We find that the $L[m^P_{t,t+1}]$ produced by the DH, RU-SV, and RU-CJI models are 0.0203, 0.0237, and 0.0197, respectively, while the data-based lower bound on $L[m^P_{t,t+1}]$ is 0.0348. The reported p-values indicate that all the three models are rejected at the 5% level, namely, the models also generate insufficient entropy $L[m^P_{t,t+1}]$. In essence, both $L[m^P_{t,t+1}]$ and $L[m_{t,t+1}]$ agree in suggesting that the three models are misspecified along the dimension of entropy bounds on the level of $m^P_{t,t+1}$ and $m_{t,t+1}$.

Elaborating further, we now argue that considering the entropy $L[(m^P_{t,t+1})^2]$ in model assessment can provide an important contrast to our findings based on the entropy $L[m^P_{t,t+1}]$. The prominent result is that the entropy $L[(m^P_{t,t+1})^2]$ of the RU-CJI model is about 15-fold higher than the other two models that do not incorporate the random jump feature in the dynamics of the consumption growth. For example, the DH, RU-SV, and RU-CJI models generate $L[(m^P_{t,t+1})^2]$ of 0.0811, 0.095, and 1.4858, respectively. Given that the lower bound restriction implied from asset prices is 0.1851, the DH and RU-SV models are rejected at the 5% level. However, the RU-CJI model with constant jump intensity cannot be rejected at the 5% level, which is a point of departure based on the entropy $L[m^P_{t,t+1}]$.

Accordingly, one key question emerges: Why does the RU-CJI fail to explain features of $m^P_{t,t+1}$ as reflected in asset prices when $L[m^P_{t,t+1}]$-based measure is used, while the model is successful in explaining features of $m^P_{t,t+1}$ as reflected in asset prices when $L[(m^P_{t,t+1})^2]$-based measure is used? To investigate a source of model outperformance, we note that the entropy measure $L[(m^P_{t,t+1})^2]$ is substantially more sensitive to tail asymmetries and tail size of the $m^P_{t,t+1}$ distribution as opposed to the entropy measure $L[m^P_{t,t+1}]$. Taking this trait of entropies into consideration, we report the moments of $m^P_{t,t+1}$ and $m_{t,t+1}$ for each of the models in Panel C of Table 3. The unexpected finding is that RU-CJI model embeds excessive levels of skewness and kurtosis of $m^P_{t,t+1}$, while generating variance that is almost 90 times its DH and RU-SV model counterparts. Our contention is that the inordinate levels of the higher-order moments of $m^P_{t,t+1}$ give rise to the reported $L[(m^P_{t,t+1})^2]$ of 1.4858 for the RU-CJI model.

How should one interpret a model, such as the RU-CJI, that calibrates well to the first-moment, the second-moment and the autocorrelation of consumption growth, but does not produce finite central moments for the distribution of both $m^P_{t,t+1}$ and $m_{t,t+1}$. This result arises because a convex transform of a random variable, which is here poisson-distributed, increases the skewness to the right (see van Zwet (1966, page 10, Theorem 2.2.1)). To see this analytically, we can invoke the density of the poisson random variable to show that $E_t[(m_{t,t+1})^k] = E_t\left[e^{k\log(m_{t,t+1})}\right] = E_t\left[E_t\left[e^{k\log(m_{t,t+1})}|j\right]\right] = e^{G[k]}E_t\left[e^{H[k]j}\right]$, for some constants $G[k]$ and $H[k]$. Note that $e^{H[k]j}$ is a convex transformation of the poisson variable J, and for certain parameterizations, does not admit finite higher-moments of $m_{t,t+1}$. The inordinate amounts of skewness and kurtosis do not appear to be a reasonable depiction of valuation operators, which are likely to be characterized by exponential, rather than power, tails.

How general are our conclusions with respect to the RU-CJI model? Specifically, are there model combinations that produce reasonable higher-order moments of $m^P_{t,t+1}$ and that calibrate well to consumption growth data, and yet deliver high entropies? To probe this issue, we vary the jump distribution parameters (θ, δ, h) of the consumption growth dynamics (see equation (36)), and report the results in Table Appendix-II. The takeaway message is that jump parameterizations (among the 27 parameter combinations) that yield reasonable levels of skewness and kurtosis of $m^P_{t,t+1}$ do not appear to produce high enough entropies to satisfy the lower bound on $L[m^P_{t,t+1}]$ and $L[(m^P_{t,t+1})^2]$.

Next we examine the entropy of $(m^T_{t,t+1})^2$ and $m^T_{t,t+1}$, which enables us to further challenge models by assessing their ability to fit certain aspects of the Treasury market data. Two features of our findings are worth emphasizing in Table 4. First, all the models fail to produce a transitory component of the SDFs that are consistent with long-term discount bond returns. Second, the RU-CJI model is worse than the other two models when performance is assessed based on the transitory component. Specifically, the jump parametrization of the consumption growth dynamics lead to even more implausible entropies of the transitory component of the RU-CJI model. Therefore, the adequate performance of the RU-CJI model, alluded to in Backus, Chernov, and Zin (2013), is also illusionary when benchmarked against our entropy-based quantification of the transitory component.

How deft are the models in matching entropy-based codependence between $m^P_{t,t+1}$ and $m^T_{t,t+1}$? Table 5 shows that the DH, RU-SV, and RU-CJI models are not able to reproduce the magnitude and the sign of the dependence measures obtained from asset prices. Although the observed asset prices indicate a positive dependence between the permanent and transitory components of the SDF, all the three models suggest a negative codependence between $m^P_{t,t+1}$ and $m^T_{t,t+1}$. Therefore, these models are not properly aligned with codependence imputed from asset market data.

In sum, for the set of parameter values in Table Appendix-I, the asset pricing models we investigate are not able to generate sufficient entropies $L[m^P_{t,t+1}]$ and $L[m_{t,t+1}]$ to describe the features of the SDF, as reflected in asset prices. Therefore, the conclusions reached elsewhere, based on the $L[m_{t,t+1}]$ measure, may be fragile to departures from the normality of $\log(m_{t,t+1})$, and hinge solely on matching the first-moment of the returns data. While the RU-CJI model does meet the lower entropy bound $L[(m^P_{t,t+1})^2]$, the model success is achieved at the expense of implausible central moments of the $m^P_{t,t+1}$ distribution. Each asset pricing model also appears inconsistent with the data on long-term bond returns, and with our entropy-based codependence measures inferred from the returns data.

7. Generalizing the entropy bounds to alternative investment horizons

The objective is to generalize the entropy bounds presented in Theorem 1 to the case when returns are measured over more than a single-period. We are also guided by Hansen (2012), who emphasizes the need to study the behavior of long-term entropy of SDFs. This problem entails imposing additional restrictions on the dynamic link between the permanent and transitory components over an n-period investment horizon.

Consider the n-period SDF, $m_{t,t+n}$, defined as:

$$m_{t,t+n} = \prod_{j=1}^{n} m_{t+j-1,t+j}, \qquad \text{where } m_{t+j-1,t+j} \text{ is the SDF from } t+j-1 \text{ to } t+j. \tag{41}$$

We postulate that the n-period SDF can be decomposed as

$$m_{t,t+n} = m^P_{t,t+n} m^T_{t,t+n}, \quad \text{where} \quad m^P_{t,t+n} = \prod_{j=1}^{n} m^P_{t+j-1,t+j} \quad \text{and} \quad m^T_{t,t+n} = \prod_{j=1}^{n} m^T_{t+j-1,t+j}, \tag{42}$$

with $m^T_{t+j-1,t+j} = 1/R_{t+j-1,t+j,\infty}$, $E[m^P_{t+j-1,t+j}] = 1$, and $R_{t+j-1,t+j,\infty}$ is the gross return from holding a discount bond with infinite-maturity from time $t+j-1$ to $t+j$.

Now consider the sets that correctly price $N+2$ assets:

$$\mathbb{S}^{(n)} = \left\{ m_{t,t+n} > 0 : E_t\left[m_{t,t+n}\right] = q_t^{(n)},\ E\left[m_{t,t+n} R_{t,t+n,\infty}\right] = 1, \text{ and } E\left[m_{t,t+n}\mathbf{R}_{t,t+n}\right] = \mathbf{1} \right\} \text{ and} \tag{43}$$

$$\mathbb{S}_P^{(n)} = \left\{ m_{t,t+n} \in \mathbb{S}^{(n)}\colon m_{t,t+n} = m^P_{t,t+n} m^T_{t,t+n}, \text{ and } m^T_{t,t+n} = \left(R_{t,t+n,\infty}\right)^{-1} \right\}, \tag{44}$$

where $\mathbf{R}_{t,t+n}$ is a vector column of risky asset returns and $q_t^{(n)}$ is the price of an n-period discount bond. Each component of $\mathbf{R}_{t,t+n}$ is of the form $R_{t,t+n} = \prod_{j=1}^{n} R_{t+j-1,t+j}$, where $R_{t+j-1,t+j}$ is the return of the risky asset from $t+j-1$ to $t+j$. Moreover, $R_{t,t+n,\infty} = \prod_{j=1}^{n} R_{t+j-1,t+j,\infty}$.

By analogy to the one-period setup, we define

$$\mathbf{y}^{(n)} = \left(\Sigma^{(n)}\right)^{-1} \left(\mathbf{1} - E[q_t^{(n)}]\, E\left[\mathbf{R}_{t,t+n}\right]\right) \quad \text{and} \quad \mathbf{y}_P^{(n)} = \left(\Sigma_P^{(n)}\right)^{-1} \left(\mathbf{1} - E\left[\mathbf{R}_{t,t+n}/R_{t,t+n,\infty}\right]\right), \tag{45}$$

where $\Sigma^{(n)}$ is the variance covariance matrix of $\mathbf{R}_{t,t+n}$ and $\Sigma_P^{(n)}$ is the variance covariance of $\mathbf{R}_{t,t+n}/R_{t,t+n,\infty}$. For parsimony of presentation, we further define (assume $(\mathbf{w}^{(n)})'\mathbf{R}_{t,t+n} > 0$):

$$\mathrm{er}_R^{(n)} \equiv E\left[\log\left(\left(\mathbf{w}^{(n)\prime}\mathbf{R}_{t,t+n}\right)^{1/n}\right)\right] - \log\left(1/E[q_t^{(1)}]\right), \quad \mathbf{w}^{(n)} \equiv \mathbf{y}^{(n)}/\left(\mathbf{1}'\mathbf{y}^{(n)}\right), \tag{46}$$

$$\mathrm{v}_R^{(n)} \equiv \log\left(1 + \mathbf{y}^{(n)\prime}\Sigma^{(n)}\mathbf{y}^{(n)} \left(E[q_t^{(n)}]\right)^{-2}\right), \qquad \mathrm{v}_P^{(n)} \equiv \log\left(1 + \mathbf{y}_P^{(n)\prime}\Sigma_P^{(n)}\mathbf{y}_P^{(n)}\right), \tag{47}$$

$$\mathrm{Q}^{(n)} \equiv \log\left(E[q_t^{(1)}]\right) - \frac{1}{n}\log\left(E[q_t^{(n)}]\right), \text{ and} \qquad \mathrm{er}_\infty^{(n)} \equiv \frac{1}{n}E\left[\log\left(R_{t,t+n,\infty}\right)\right] - \log\left(1/E[q_t^{(1)}]\right). \tag{48}$$

Our main characterization is presented next.

Theorem 3 *The following entropy bounds are applicable to n-period stochastic discount factors:*

(a) The entropy of $\left(m^P_{t,t+n}\right)^2$ *and* $m^P_{t,t+n}$ *satisfy:*

$$L[\left(m^P_{t,t+n}\right)^2] \geq \mathrm{v}^{(n)}_P \; + \; 2n\left(\mathrm{er}^{(n)}_R - \mathrm{er}^{(n)}_\infty\right) \qquad \text{and} \qquad L[m^P_{t,t+n}] \geq n\left(\mathrm{er}^{(n)}_R - \mathrm{er}^{(n)}_\infty\right). \tag{49}$$

(b) The entropy of $m^2_{t,t+n}$ *and* $m_{t,t+n}$ *satisfy:*

$$L\left[m^2_{t,t+n}\right] \geq \mathrm{v}^{(n)}_R \; + \; 2n\left(\mathrm{er}^{(n)}_R - \mathrm{Q}^{(n)}\right) \qquad \text{and} \qquad L\left[m_{t,t+n}\right] \geq n\left(\mathrm{er}^{(n)}_R - \mathrm{Q}^{(n)}\right). \tag{50}$$

Proof: See Online Appendix V. ■

The entropy bounds derived in Theorem 3 reflect information about the dynamics of returns and the Treasury yield curve. Our entropy restrictions on $m^P_{t,t+n}$ and $m_{t,t+n}$ can be used to evaluate consistency of asset pricing models with observed prices over any investment horizon.

Other forms of codependence could be clarified in a multi-period setting, whereby $L[m^P_{t,t+n}\, m^T_{t,t+n}] - L[m^P_{t,t+n}] - L[m^T_{t,t+n}] = \log(E[q^{(n)}_t]) - \log\left(E\left[1/R_{t,t+n,\infty}\right]\right)$. Elaborating further, the dependence between $m_{t,t+k}$ and $m_{t+k,t+n}$ can be expressed in terms of the Treasury yield curve quantities as: $L[m_{t,t+k} m_{t+k,t+n}] - L\left[m_{t,t+k}\right] - L[m_{t+k,t+n}] = \log(E[q^{(n)}_t]) - \log(E[q^{(k)}_t]) - \log(E[q^{(n-k)}_{t+k}])$.

Overall, the restrictions over the n-periods could enrich our understanding of the codependence between the permanent and the transitory components of the SDF and help to build models that are more adept at mimicking asset pricing quantities over alternative investment horizons.

8. Conclusions

A central problem in finance is the specification of the stochastic discount factor. We study this problem by providing new asset pricing restrictions that are based on the entropy of the square of the permanent component of the stochastic discount factor. Our entropy measure is suitable for capturing the conditional volatility of the log permanent component of the stochastic discount factor and also non-normalities in the log permanent component. The entropy restrictions we develop are based on the ability of the stochastic discount factor to correctly price the risk-free bond, the long-term discount bond, and a set of risky assets. We also present new entropy codependence measures to assess asset pricing models.

Our bounds framework hinges on understanding the permanent and transitory components of the stochastic discount factors and are in the tradition of Alvarez and Jermann (2005), Hansen and Scheinkman (2009), Bakshi and Chabi-Yo (2012), and Hansen (2012). Key to our analysis are the expressions for the permanent and the transitory components of the stochastic discount factor, which we obtain by solving the eigenfunction problem.

There are a number of implications of our entropy framework for asset pricing models. First, our evaluation reveals that the difference habit model, the recursive utility model with stochastic variance, and the recursive utility model with constant jump intensity generally fail to satisfy the posited bounds on the permanent and the transitory components of the stochastic discount factors. Second, while the recursive utility model with constant jump intensity meets the lower bound on the square of the permanent component, we attribute the model success to unrealistic higher-order moments associated with the parametrization of the stochastic discount factor. Finally, these models are incompatible with the entropy co-dependency restrictions inferred from the returns data.

We also extend our framework to bounds that are valid for stochastic discount factors over alternative investment horizons. Borovicka, Hansen, Hendricks, and Scheinkman (2011) have advocated looking at risk and valuation dynamics over different investment horizons.

With some modifications, our framework could be expanded to analyze other asset pricing models, including generalized recursive smooth ambiguity utility (as in Ju and Miao (2012)) and generalized disappointment aversion (as in Routledge and Zin (2010)). One could also refine our bounds framework to incorporate conditioning information to further learn about the properties of the stochastic discount factors and the dynamics of the permanent and transitory components.

The push to attain well-specified stochastic discount factors has applications that transcend stock, bond, commodity, currency, and options valuation.

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Appendix A: Proof of Theorem 1

Proof of the entropy bound on $(m^P_{t,t+1})^2$. To derive the bound in equation (21) of Theorem 1, we write

$$\begin{aligned} L[(m^P_{t,t+1})^2] &= \log(E[(m^P_{t,t+1})^2]) - E[\log((m^P_{t,t+1})^2)], \\ &= \log(E[(m^P_{t,t+1})^2]) - 2E[\log(m^P_{t,t+1})], \\ &= \log(E[(m^P_{t,t+1})^2]) + 2L\left[m^P_{t,t+1}\right], \\ &= \log\left(1 + Var\left[m^P_{t,t+1}\right]\right) + 2L[m^P_{t,t+1}]. \end{aligned} \quad \text{(A1)}$$

Since $L[m^P_{t,t+1}] \geq \text{er}_R - \text{er}_\infty$ (shown in equation (A15)), we deduce

$$L[\left(m^P_{t,t+1}\right)^2] \;\geq\; \log\left(1 + Var\left[m^P_{t,t+1}\right]\right) \;+\; 2\left(\text{er}_R - \text{er}_\infty\right). \quad \text{(A2)}$$

Since $E\left[m_{t,t+1}\mathbf{R}_{t,t+1}\right] = E\left[m^P_{t,t+1}\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}}\right] = \mathbf{1}$, we then obtain:

$$E\left[m^P_{t,t+1}\left(\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}} - E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}}\right]\right)\right] = \mathbf{1} - E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}}\right]. \quad \text{(A3)}$$

Multiplying each side of equation (A3) by $\left(\mathbf{1} - E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1\infty}}\right]\right)' \Sigma_P^{-1}$, we get

$$\left(\mathbf{1} - E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1\infty}}\right]\right)' \Sigma_P^{-1}\left(\mathbf{1} - E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}}\right]\right) = E\left[m^P_{t,t+1}\left(\mathbf{1} - E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1\infty}}\right]\right)' \Sigma_P^{-1}\left(\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}} - E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}}\right]\right)\right]. \quad \text{(A4)}$$

Applying the Cauchy Schwartz inequality to the right-hand side of equation (A4), we note that

$$\begin{aligned} Var\left[m^P_{t,t+1}\right] &\geq \left(\mathbf{1} - E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1\infty}}\right]\right)' \Sigma_P^{-1}\left(\mathbf{1} - E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1\infty}}\right]\right), \\ &\geq \left(\mathbf{1} - E\left[\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1\infty}}\right]\right)' (\Sigma_P^{-1})' \Sigma_P \Sigma_P^{-1}\left(\mathbf{1} - E\left(\frac{\mathbf{R}_{t,t+1}}{R_{t,t+1\infty}}\right)\right), \\ &\geq \mathbf{y}_P' \Sigma_P \mathbf{y}_P. \qquad \text{(where setting } \mathbf{y}_P \equiv \Sigma_P^{-1}\left(\mathbf{1} - E\left[\mathbf{R}_{t,t+1}/R_{t,t+1,\infty}\right]\right)) \end{aligned} \quad \text{(A5)}$$

Hence, equation (A2) reduces to

$$L[\left(m^P_{t,t+1}\right)^2] \;\geq\; 2\left(\text{er}_R - \text{er}_\infty\right) \;+\; \log\left(1 + \mathbf{y}_P' \Sigma_P \mathbf{y}_P\right). \quad \text{(A6)}$$

This was the desired final step. ■

Proof of the entropy bound on $m^2_{t,t+1}$. By the definition of entropy: $L[m^2] = \log(E[m^2]) - E[\log(m^2)]$. Then

$$\begin{aligned}
L\left[m^2_{t,t+1}\right] &= \log\left(E\left[m^2_{t,t+1}\right]\right) - 2\log\left(E\left[q_t\right]\right) + 2L\left[m_{t,t+1}\right], \\
&= \log\left(1 + \frac{E\left[m^2_{t,t+1}\right] - \left(E\left[q_t\right]\right)^2}{\left(E\left[q_t\right]\right)^2}\right) + 2L\left[m_{t,t+1}\right], \\
&= \log\left(1 + \frac{Var\left[m_{t,t+1}\right]}{\left(E\left[q_t\right]\right)^2}\right) + 2L\left[m_{t,t+1}\right], \\
&\geq \log\left(1 + \frac{Var\left[m_{t,t+1}\right]}{\left(E\left[q_t\right]\right)^2}\right) + 2\,\mathrm{er}_R \qquad \text{(since } L\left[m_{t,t+1}\right] \geq \mathrm{er}_R\text{; see } (A19)\text{)}. \qquad \text{(A7)}
\end{aligned}$$

Because $E\left[m_{t,t+1}\mathbf{R}_{t,t+1}\right] = \mathbf{1}$ and setting $q_t = E\left[m_{t,t+1}\right]$, it follows that

$$E\left[m_{t,t+1}\left(\mathbf{R}_{t,t+1} - E\left(\mathbf{R}_{t,t+1}\right)\right)\right] = \left(\mathbf{1} - E\left[q_t\right]E\left[\mathbf{R}_{t,t+1}\right]\right). \qquad \text{(A8)}$$

Multiplying equation (A8) by $\left(\mathbf{1} - \left(E\left[q_t\right]\right)E\left[\mathbf{R}_{t,t+1}\right]\right)' \Sigma^{-1}$ yields

$$\left(\mathbf{1} - \left(E\left[q_t\right]\right)E\left[\mathbf{R}_{t,t+1}\right]\right)' \Sigma^{-1}\left(\mathbf{1} - E\left[q_t\right]E\left[\mathbf{R}_{t,t+1}\right]\right) = E\left[m_{t,t+1}\left(\mathbf{1} - \left(E\left[q_t\right]\right)E\left[\mathbf{R}_{t,t+1}\right]\right)' \Sigma^{-1}\left(\mathbf{R}_{t,t+1} - E\left[\mathbf{R}_{t,t+1}\right]\right)\right]. \qquad \text{(A9)}$$

Applying the Cauchy Schwartz inequality to the right-hand side of equation (A9), it can be shown that

$$\begin{aligned}
Var\left[m_{t,t+1}\right] &\geq \left(\mathbf{1} - \left(E\left[q_t\right]\right)E\left[\mathbf{R}_{t,t+1}\right]\right)' \Sigma^{-1}\left(\mathbf{1} - \left(E\left[q_t\right]\right)E\left[\mathbf{R}_{t,t+1}\right]\right), \\
&\geq \left(\mathbf{1} - \left(E\left[q_t\right]\right)E\left[\mathbf{R}_{t,t+1}\right]\right)' \Sigma^{-1}\Sigma\Sigma^{-1}\left(\mathbf{1} - \left(E\left[q_t\right]\right)E\left[\mathbf{R}_{t,t+1}\right]\right), \\
&\geq \mathbf{y}'\Sigma\mathbf{y}. \qquad \text{(A10)}
\end{aligned}$$

Hence, we obtain

$$L\left[m^2_{t,t+1}\right] \geq 2\,\mathrm{er}_R + \mathrm{v}_R, \qquad \text{where defining} \qquad \mathrm{v}_R \equiv \log\left(1 + \frac{\mathbf{y}'\Sigma\mathbf{y}}{\left(E\left[q_t\right]\right)^2}\right). \qquad \text{(A11)}$$

This completes the description of the proof. ■

Generalizing the Alvarez and Jermann (2005) entropy bound on $m^P_{t,t+1}$ to many risky assets. Consider an SDF $m^P_{t,t+1} \in \mathbb{S}_P$. We note that

$$E\left[\log\left(m_{t,t+1}\mathbf{w}^{'}\mathbf{R}_{t,t+1}\right)\right] = E\left[\log\left(m^P_{t,t+1}\frac{\mathbf{w}^{'}\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}}\right)\right]. \tag{A12}$$

Invoking Jensen's inequality, we have

$$\begin{aligned} E\left[\log\left(m^P_{t,t+1}\frac{\mathbf{w}^{'}\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}}\right)\right] &= E\left[\log\left(m_{t,t+1}\mathbf{w}^{'}\mathbf{R}_{t,t+1}\right)\right], \\ &\leq \log\left(\mathbf{w}^{'}E\left[m_{t,t+1}\mathbf{R}_{t,t+1}\right]\right), \\ &\leq \log\left(\mathbf{w}^{'}\mathbf{1}\right), \\ &\leq 0. \end{aligned} \tag{A13}$$

From equation (A13), we deduce

$$E\left[\log\left(\frac{\mathbf{w}^{'}\mathbf{R}_{t,t+1}}{R_{t,t+1,\infty}}\right)\right] \leq -E[\log\left(m^P_{t,t+1}\right)] = L[m^P_{t,t+1}]. \tag{A14}$$

Hence,

$$\begin{aligned} L\left[m^P_{t,t+1}\right] &\geq E[\log(\mathbf{w}^{'}\mathbf{R}_{t,t+1})] - E[\log(R_{t,t+1,\infty})], \\ &\geq \underbrace{E[\log(\mathbf{w}^{'}\mathbf{R}_{t,t+1})] - \log\left(1/E\left[q_t\right]\right)}_{\equiv \text{er}_R} - \underbrace{\left(E[\log(R_{t,t+1,\infty})] - \log\left(1/E\left[q_t\right]\right)\right)}_{\equiv \text{er}_\infty}. \end{aligned} \tag{A15}$$

This bound generalizes Alvarez and Jermann (2005) to $N+2$ assets. ■

Generalizing the Backus, Chernov, and Zin (2013) entropy bound on $m_{t,t+1}$ to many risky assets.

$$\begin{aligned} E\left[\log\left(m_{t,t+1}\mathbf{w}^{'}\mathbf{R}_{t,t+1}\right)\right] &\leq \log\left(E\left[m_{t,t+1}\mathbf{w}^{'}\mathbf{R}_{t,t+1}\right]\right), \\ &\leq \log\left(\mathbf{w}^{'}E\left[m_{t,t+1}\mathbf{R}_{t,t+1}\right]\right), \\ &\leq \log\left(\mathbf{w}^{'}\mathbf{1}\right) = \log\left(1\right), \\ &\leq 0. \end{aligned} \tag{A16}$$

From equation (A16), we deduce

$$E\left[\log\left(\mathbf{w}^{'}\mathbf{R}_{t,t+1}\right)\right] \leq -E\left[\log\left(m_{t,t+1}\right)\right]. \tag{A17}$$

Adding $\log\left(E\left[m_{t,t+1}\right]\right)$ to both sides of equation (A17) yields

$$\log\left(E\left[m_{t,t+1}\right]\right)+E[\log(\mathbf{w}^{'}\mathbf{R}_{t,t+1})] \;\leq\; \log(E[m_{t,t+1}])-E\left[\log\left(m_{t,t+1}\right)\right]=L\left[m_{t,t+1}\right]. \tag{A18}$$

Since $q_t = E_t\left[m_{t,t+1}\right]$, equation (A18) simplifies to

$$L\left[m_{t,t+1}\right] \geq \mathrm{er}_R, \qquad \text{where defining} \qquad \mathrm{er}_R = E[\log(\mathbf{w}^{'}\mathbf{R}_{t,t+1})] \;-\; \log\left(1/E\left[q_t\right]\right). \tag{A19}$$

Our equation (A19) generalizes Backus, Chernov, and Zin (2013) when the bounds incorporate more than a single risky asset, specifically the set of assets outlined in equation (16). ■

Appendix B: Proof of Theorem 2 on codependence

To streamline expressions, we write our measure of codependence as:

$$D_b \;\equiv\; L[(m^P_{t,t+1})^2(m^T_{t,t+1})^2] \;-\; L[(m^P_{t,t+1})^2] \;-\; L[(m^T_{t,t+1})^2], \tag{B1}$$

$$\;=\; \log(E[(m^P_{t,t+1})^2(m^T_{t,t+1})^2])-\log(E[(m^P_{t,t+1})^2])-\log(E[(m^T_{t,t+1})^2]). \tag{B2}$$

From the expression in equation (B1),

$$\log(E[(m^P_{t,t+1})^2])+\log(E[(m^T_{t,t+1})^2])=\log(E[(m^P_{t,t+1})^2(m^T_{t,t+1})^2]) \;-\; D_b. \tag{B3}$$

From the Cauchy Schwartz inequality, we have

$$\left(E\left[m^P_{t,t+1}m^T_{t,t+1}\right]\right)^2 \;\leq\; E[(m^P_{t,t+1})^2]E[(m^T_{t,t+1})^2]. \tag{B4}$$

Taking the log of the expression in equation (B1) gives

$$\log(\left(E[m^P_{t,t+1}m^T_{t,t+1}]\right)^2)\leq\log(E[\left(m^P_{t,t+1}\right)^2])+\log(E[\left(m^T_{t,t+1}\right)^2]). \tag{B5}$$

Replacing equation (B3) in the expression (B5) yields

$$\log((E[q_t])^2)\leq\log(E[\left(m^P_{t,t+1}\right)^2(m^T_{t,t+1})^2]) \;-\; D_b, \qquad (\text{since } \; q_t=E_t\left[m^P_{t,t+1}m^T_{t,t+1}\right]) \tag{B6}$$

and

$$D_b \;+\; \log((E\,[q_t])^2) \;\leq\; \log\left(E[(m^P_{t,t+1})^2\left(m^T_{t,t+1}\right)^2]\right). \tag{B7}$$

Since $Var[m_{t,t+1}] = E[m^2_{t,t+1}] - (E[m_{t,t+1}])^2 = E[(m^P_{t,t+1})^2(m^T_{t,t+1})^2] - (E\,[q_t])^2$, one could write equation (B7) as:

$$D_b + \log((E\,[q_t])^2) \leq \log(Var\,[m_{t,t+1}] + (E\,[q_t])^2). \tag{B8}$$

From the proof of Theorem 1, we have shown that $\mathbf{y}'\,\Sigma\mathbf{y} \;\leq\; Var\,[m_{t,t+1}]$. Therefore,

$$\log\left(\mathbf{y}'\Sigma\mathbf{y} \;+\; (E\,[q_t])^2\right) \;\leq\; \log\left(Var\,[m_{t,t+1}] + (E\,[q_t])^2\right). \tag{B9}$$

Because $\mathbf{y}'\Sigma\mathbf{y}$ is the highest lower bound on $Var\,[m_{t,t+1}]$, it follows that $\log(\mathbf{y}'\Sigma\mathbf{y} + (E\,[q_t])^2)$ is the highest lowest bound on $\log(Var\,[m_{t,t+1}] + (E\,[q_t])^2)$. Therefore, any other lower bound on the quantity $\log(Var\,[m_{t,t+1}] + (E\,[q_t])^2)$ must be lower than $\log(\mathbf{y}'\Sigma\mathbf{y} + (E\,[q_t])^2)$. As a result,

$$D_b + \log\left(\left(E\left[q^1_t\right]\right)^2\right) \;\leq\; \log\left(\mathbf{y}'\,\Sigma\,\mathbf{y} + (E\,[q_t])^2\right), \tag{B10}$$

which implies

$$D_b \;\leq\; \log\left(1 + \frac{\mathbf{y}'\,\Sigma\,\mathbf{y}}{(E\,[q_t])^2}\right). \tag{B11}$$

To establish the positivity of the codependence measure D_b, we note that

$$D_b \;=\; \log\left(\frac{E\left[\left(m^P_{t,t+1}\right)^2\left(m^T_{t,t+1}\right)^2\right]}{E\left[\left(m^P_{t,t+1}\right)^2\right]E\left[\left(m^T_{t,t+1}\right)^2\right]}\right), \tag{B12}$$

$$\;=\; \log\left(\frac{Cov\left[\left(m^P_{t,t+1}\right)^2,\left(m^T_{t,t+1}\right)^2\right]}{E\left[\left(m^P_{t,t+1}\right)^2\right]E\left[\left(m^T_{t,t+1}\right)^2\right]} + 1\right). \tag{B13}$$

Since $Cov[m^P_{t,t+1}, m^T_{t,t+1}] = E\,[q_t] - E\,[1/R_{t,t+1,\infty}] > 0$, we have $Cov[(m^P_{t,t+1})^2,(m^T_{t,t+1})^2] > 0$. Therefore, $D_b > \log(1) = 0$. The proof of Theorem 2 is complete. ■

Online Appendix I: Proofs of the results in Section 3

Proof of equation (6) in Example 1. We apply the definition of entropy to $(m^P_{t,t+1})^2$, i.e., $L[(m^P_{t,t+1})^2] = \log(E[(m^P_{t,t+1})^2]) - E[\log((m^P_{t,t+1})^2)]$. Under our assumption, $\log(m^P_{t,t+1})$ is normally distributed with mean

$-\frac{1}{2}\sigma_P^2$ and variance σ_P^2. Therefore,

$$L[m^P_{t,t+1}] = -E\left[\log\left(m^P_{t,t+1}\right)\right] = \frac{1}{2}\sigma_P^2, \tag{C1}$$

and,

$$L[(m^P_{t,t+1})^2] = \log\left(\exp\left(-\frac{2}{2}\sigma_P^2 + \frac{4}{2}\sigma_P^2\right)\right) \; - \; 2\left(-\frac{1}{2}\sigma_P^2\right) = 2\sigma_P^2. \tag{C2}$$

Using equations (C1) and (C2), we see that $L[(m^P_{t,t+1})^2] = 4L[m^P_{t,t+1}]$. ■

Proof of equation (7). Using the definition of entropy

$$\begin{aligned} L[(m^P_{t,t+1})^2] - 4L[m^P_{t,t+1}] &= \log(E[\left(m^P_{t,t+1}\right)^2]) - 2E[\log\left(m^P_{t,t+1}\right)] + 4E[\log(m^P_{t,t+1})], & \text{(C3)} \\ &= \log(E[\left(m^P_{t,t+1}\right)^2]) + E[\log\left(((m^P_{t,t+1})^2\right)]. & \text{(C4)} \end{aligned}$$

Thus, we have the desired expression. ■

Proof of equation (8) in Example 2. Combining our assumption that $\log(m^P_{t,t+1})$ follows $\mathcal{N}\left(\mu_t, \sigma_t^2\right)$ with the fact that $E_t[m^P_{t,t+1}] = 1$, we note

$$e^{\mu_t + \frac{1}{2}\sigma_t^2} = 1, \text{ which implies } \mu_t = -\frac{1}{2}\sigma_t^2. \text{ Hence, } L[m^P_{t,t+1}] = \frac{1}{2}E[\sigma_t^2]. \tag{C5}$$

Next, we evaluate $L[(m^P_{t,t+1})^2] = \log(E[(m^P_{t,t+1})^2]) - E[\log((m^P_{t,t+1})^2)]$ in two steps. Since $\log(m^P_{t,t+1})$ follows $\mathcal{N}\left(\mu_t, \sigma_t^2\right)$, we obtain

$$E[(m^P_{t,t+1})^2] = E[E_t[\left(m^P_{t,t+1}\right)^2]] = E[e^{2\mu_t + 2\sigma_t^2}] = E[e^{\sigma_t^2}] \qquad (\text{since } \mu_t = -\frac{1}{2}\sigma_t^2). \tag{C6}$$

With the above results, we note that $L[(m^P_{t,t+1})^2] = \log(E[(m^P_{t,t+1})^2]) - E[\log((m^P_{t,t+1})^2)] = \log(E[e^{\sigma_t^2}]) - 2\mu_t = \log(E[e^{\sigma_t^2}]) + E[\sigma_t^2]$. ■

Proof of equation (9) in Example 2. Observe that

$$L[(m^P_{t,t+1})^2] - 4L[m^P_{t,t+1}] = \log(E[e^{\sigma_t^2}]) + E[\sigma_t^2] - 4\left(\frac{1}{2}E\left[\sigma_t^2\right]\right) = \log\left(\frac{E\left[e^{\sigma_t^2}\right]}{\exp\left(E\left[\sigma_t^2\right]\right)}\right). \tag{C7}$$

Taylor expansion of $e^{\sigma_t^2}$ around $\sigma_t^2 = E\left[\sigma_t^2\right]$ yields

$$e^{\sigma_t^2} = e^{\left(E\left[\sigma_t^2\right]\right)} + \sum_{j=1}^{\infty} \frac{1}{j!}\left(\sigma_t^2 - E\left[\sigma_t^2\right]\right)^j e^{\left(E\left[\sigma_t^2\right]\right)}, \tag{C8}$$

which implies

$$\frac{E\left[e^{\sigma_t^2}\right]}{e^{\left(E\left[\sigma_t^2\right]\right)}} = 1 + \sum_{j=1}^{\infty} \frac{1}{j!} E\left[\left(\sigma_t^2 - E\left[\sigma_t^2\right]\right)^j\right] = 1 + \sum_{j=2}^{\infty} \frac{1}{j!} E\left[\left(\sigma_t^2 - E\left[\sigma_t^2\right]\right)^j\right]. \tag{C9}$$

Therefore,

$$L[(m_{t,t+1}^P)^2] - 4L[m_{t,t+1}^P] = \log\left(\frac{E\left[e^{\sigma_t^2}\right]}{e^{\left(E\left[\sigma_t^2\right]\right)}}\right) = \log\left(1 + \sum_{j=2}^{\infty} \frac{1}{j!} E\left[\left(\sigma_t^2 - E\left[\sigma_t^2\right]\right)^j\right]\right). \tag{C10}$$

This ends the proof of equation (9). ■

Proof of equation (10). We observe that

$$\exp\left(\log((m_{t,t+1}^P)^2)\right) = \exp\left(2\log\left(m_{t,t+1}^P\right)\right). \tag{C11}$$

Taylor expansion series of $\exp\left(\log((m_{t,t+1}^P)^2)\right)$ around $E[\log(m_{t,t+1}^P)]$ produces

$$\exp\left(2\log\left(m_{t,t+1}^P\right)\right) = \exp\left(2E\left[\log\left(m_{t,t+1}^P\right)\right]\right) + \sum_{j=1}^{\infty} \frac{2^j}{j!}\left(\log\left(m_{t,t+1}^P\right) - E\left[\log\left(m_{t,t+1}^P\right)\right]\right)^j. \tag{C12}$$

We apply the expectation operator to (C12) and get

$$E\left[\exp\left(2\log\left(m_{t,t+1}^P\right)\right)\right] = \exp\left(2E\left[\log\left(m_{t,t+1}^P\right)\right]\right) + \sum_{j=1}^{\infty} \frac{2^j}{j!}\kappa_j \exp\left(2E\left[\log\left(m_{t,t+1}^P\right)\right]\right), \tag{C13}$$

with $\kappa_j = E\left[\left(\log\left(m_{t,t+1}^P\right) - E\left[\log\left(m_{t,t+1}^P\right)\right]\right)^j\right]$. Next, we apply the log function to (C13) and get

$$\log\left(E\left[\exp\left(2\log\left(m_{t,t+1}^P\right)\right)\right]\right) = \log\left(\exp\left(2E\left[\log\left(m_{t,t+1}^P\right)\right]\right)\right) + \log\left(1 + \sum_{j=1}^{\infty} \frac{2^j}{j!}\kappa_j\right), \tag{C14}$$

and

$$\log\left(E\left[\exp\left(2\log\left(m_{t,t+1}^P\right)\right)\right]\right) - 2E\left[\log\left(m_{t,t+1}^P\right)\right] = \log\left(1 + \sum_{j=1}^{\infty} \frac{2^j}{j!}\kappa_j\right). \tag{C15}$$

Expression (C15) is equivalent to

$$L[\left(m^P_{t,t+1}\right)^2] = \log(E[(m^P_{t,t+1})^2]) - E[\log((m^P_{t,t+1})^2)] = \log\left(1+\sum_{j=1}^{\infty}\frac{2^j}{j!}\kappa_j\right). \tag{C16}$$

This completes our description of the steps. ■

Proof of equation (11). Under the normality of $\log(m^P_{t,t+1})$, we get

$$\begin{aligned} L[(m^P_{t,t+1})^2] &= \log\left(1+2Var\left[\log\left(m^P_{t,t+1}\right)\right]\right) \\ &= \log\left(1-4E\left[\log\left(m^P_{t,t+1}\right)\right]\right) \quad (\text{since } E\left[\log\left(m^P_{t,t+1}\right)\right] = -\frac{1}{2}Var\left[\log\left(m^P_{t,t+1}\right)\right]) \\ &= -4E\left[\log\left(m^P_{t,t+1}\right)\right] = 4L[m^P_{t,t+1}], \end{aligned} \tag{C17}$$

as desired. ■

Proof of equation (12). Observe that

$$\begin{aligned} L[(m^P_{t,t+1})^2] - 2L[m^P_{t,t+1}] &= \log(E[(m^P_{t,t+1})^2]) - 2E[\log(m^P_{t,t+1})] - 2\log(E[m^P_{t,t+1}]) + 2E\left[\log\left(m^P_{t,t+1}\right)\right] \\ &= \log\left(E\left[\left(m^P_{t,t+1}\right)^2\right]\right) \\ &= \log\left(E[(m^P_{t,t+1})^2] - (E[m^P_{t,t+1}])^2 + 1\right) \\ &= \log\left(1+Var\left[m^P_{t,t+1}\right]\right) \approx Var\left[m^P_{t,t+1}\right], \end{aligned} \tag{C18}$$

which is what we present in the main body of the paper. ■

Online Appendix II: Analytical solution for the eigenfunction problem in Example 3

Consider the eigenfunction problem for the dynamics of the SDF in equation (13):

$$E_t\left[m_{t,t+1}\, e_{t+1}\right] = \zeta\, e_t, \quad \text{where } \zeta \text{ is the eigenvalue and } e_{t+1} \text{ is the eigenfunction.} \tag{D1}$$

Accordingly, the permanent and transitory components of the SDF are

$$m^P_{t,t+1} = m_{t,t+1}\left(\frac{e_{t+1}}{\zeta e_t}\right) \qquad \text{and} \qquad m^T_{t,t+1} = \frac{\zeta e_t}{e_{t+1}}. \tag{D2}$$

We conjecture that the eigenfunction e_{t+1} takes the form $e_{t+1} = \exp(\xi z_{t+1})$. Consider the expression

$$\begin{aligned}
\log(m_{t,t+1}) + \log(e_{t+1}) - \log(e_t) &= -\delta - \gamma z_t - \lambda z_t^{\frac{1}{2}} \varepsilon_{t+1} + \xi z_{t+1} - \xi z_t, \\
&= -\delta + \xi(1-\varphi)\theta + (-\gamma + \xi\varphi - \xi) z_t + (-\lambda + \xi\sigma) z_t^{\frac{1}{2}} \varepsilon_{t+1}, \quad \text{(D3)}
\end{aligned}$$

and, thus,

$$E_t\left[m_{t,t+1}\frac{e_{t+1}}{e_t}\right] = \exp\left(-\delta + \xi(1-\varphi)\theta + \left(-\gamma + \xi\varphi - \xi + \frac{1}{2}(-\lambda+\xi\sigma)^2\right) z_t\right). \quad \text{(D4)}$$

Therefore,

$$\log(\zeta) = -\delta + \xi(1-\varphi)\theta \qquad \text{and} \qquad -\gamma + \xi\varphi - \xi + \frac{1}{2}(-\lambda+\xi\sigma)^2 = 0. \quad \text{(D5)}$$

It may be seen that the second expression in equation (D5) is amenable to the simplification:

$$\frac{1}{2}\lambda^2 - \gamma + \xi(\varphi - 1 - \lambda\sigma) + \frac{1}{2}\xi^2\sigma^2 = 0. \quad \text{(D6)}$$

To be consistent with Backus, Foresi, and Telmer (2001, Section II.B), we must have $\gamma = \frac{1}{2}\left(1+\lambda^2\right)$. Let $\Delta = (\varphi - 1 - \lambda\sigma)^2 - 2\sigma^2\left(\frac{1}{2}\lambda^2 - \gamma\right) > 0$. Following Hansen and Scheinkman (2009), we select the solution associated with the negative root. Consequently, we choose

$$\xi = \frac{-\xi(\varphi - 1 - \lambda\sigma) - \sqrt{\Delta}}{\sigma^2}. \quad \text{(D7)}$$

The transitory component of the SDF is $m^T_{t,t+1} = \exp(-\delta + \xi(1-\varphi)\theta + \xi(z_t - z_{t+1}))$. Hence, the log permanent component of the SDF is $\log(m^P_{t,t+1}) = \log(m_{t,t+1}) - \log(m^T_{t,t+1})$, which delivers equation (14). The entropies in (13) follow by exploiting the conditional expectation. ■

Online Appendix III: Details of the difference habit model in Proposition 1

For the law of motions of the habit and consumption growth in equation (30), we define the backshift operators $\eta[B]$ and $\gamma[B]$:

$$\eta[B] = \sum_{j=0}^{\infty} \eta_j B^j \qquad \text{and} \qquad \gamma[B] = \sum_{j=0}^{\infty} \gamma_j B^j, \quad \text{(E1)}$$

with $\eta_0 = 1-\varphi_h$ and $\eta_{j+1} = \varphi_h \eta_j$, $j \geq 0$, and $\gamma_0 = 1$. Invoking a log linear approximation of $\log(s_t)$,

$$\log(m_{t,t+1}) = D_0 + (\rho-1)\frac{1}{s}\left(1-(1-s)\eta[B]B\right)\gamma[B]\upsilon^{\frac{1}{2}}\omega_{gt+1}, \tag{E2}$$

$$\text{where} \;\; D_0 = \log(\beta) + (\rho-1)\log(g) + (\rho-1)\frac{(s-1)}{s}\left(\frac{\eta_0}{1-\varphi_h}-1\right)\log(g).$$

Using a log linear approximation $\log(s_t) \approx 1 + \frac{(s-1)}{s}z_t$, the dynamics of the surplus consumption ratio is

$$\log(s_{t+1}) - \log(s_t) = \frac{(s-1)}{s}\left(\eta[B]B-1\right)\log(g_{t+1}). \tag{E3}$$

Therefore, we may write the log SDF as

$$\begin{aligned}\log(m_{t,t+1}) \;\; &= \;\; \log(\beta) + (\rho-1)\log(g) + (\rho-1)\frac{(s-1)}{s}\left(\eta[B]B-1\right)\log(g) \\ &\quad + (\rho-1)\frac{1}{s}\left(1-(1-s)\eta[B]B\right)\gamma[B]\upsilon^{\frac{1}{2}}\omega_{gt+1}.\end{aligned} \tag{E4}$$

To solve for the permanent and transitory components of the SDF, we write the log SDF as

$$\begin{aligned}\log(m_{t,t+1}) \;\; &= \;\; \log(\beta) + (\rho-1)\log(g) + (\rho-1)\frac{(s-1)}{s}\left(\eta[B]B-1\right)\log(g) + (\rho-1)\frac{1}{s}x_t \\ &\quad - (\rho-1)\frac{1}{s}(1-s)\eta[B]Bx_t - (\rho-1)\frac{1}{s}(1-s)\eta[B]B\upsilon^{\frac{1}{2}}\omega_{gt+1} \\ &\quad + (\rho-1)\frac{1}{s}\upsilon^{\frac{1}{2}}\omega_{gt+1},\end{aligned} \tag{E5}$$

where,

$$x_t = (\gamma[B]-\gamma_0)\upsilon^{\frac{1}{2}}\omega_{gt+1}, \qquad \text{implying} \qquad x_{t+1} - \varphi_g x_t = \gamma_1 \upsilon^{1/2}\omega_{gt+1}. \tag{E6}$$

We simplify the log SDF as

$$\begin{aligned}\log(m_{t,t+1}) \;\; &= \;\; D_0 + (\rho-1)\frac{1}{s}\left(1-\frac{1}{\gamma_1}\varphi_g\right)x_t + (\rho-1)\frac{1}{s\gamma_1}x_{t+1} \\ &\quad + (\rho-1)\frac{1}{s}(1-s)\left(\frac{1}{\gamma_1}\varphi_g - 1\right)\eta[B]x_{t-1} - (\rho-1)\frac{1}{s}(1-s)\frac{1}{\gamma_1}\eta[B]x_t.\end{aligned} \tag{E7}$$

We conjecture that the eigenfunction e_{t+1} corresponding to the general problem in equations (D1) and (D2) is of the form:

$$\log(e_{t+1}) = \delta[B]\,x_{t+1}, \qquad \text{where} \qquad \delta[B] = \sum_{j=0}^{\infty} \delta_j B^j \qquad \text{with} \qquad \delta_0 = 1. \tag{E8}$$

To verify the solution, we expand to the following:

$$\begin{aligned}
\log(m_{t,t+1}) + \log\left(\frac{e_{t+1}}{e_t}\right) &= D_0 + (\rho-1)\frac{1}{s}\left(1 - \frac{1}{\gamma_1}\varphi_g\right)x_t + (\rho-1)\frac{1}{s}(1-s)\left(\frac{1}{\gamma_1}\varphi_g - 1\right)\eta[B]\,x_{t-1} \\
&- \left((\rho-1)\frac{1}{s}(1-s)\frac{1}{\gamma_1}\eta[B] + \delta[B]\right)x_t + (\delta[B]-\delta_0)\,x_{t+1} \\
&+ \left((\rho-1)\frac{1}{s}\frac{1}{\gamma_1} + \delta_0\right)x_{t+1}.
\end{aligned} \tag{E9}$$

Upon simplifying the expectation involving the eigenfunction problem, we derive ζ as

$$\begin{aligned}
\log(\zeta) &= \log(\beta) + (\rho-1)\log(g) + (\rho-1)\frac{(s-1)}{s}(\eta[B]B - 1)\log(g) + \frac{1}{2}\left((\rho-1)\frac{1}{s}\frac{1}{\gamma_1} + \delta_0\right)^2 \gamma_1^2 \upsilon \\
&+ \left((\rho-1)\frac{1}{s}\left(1 - \frac{1}{\gamma_1}\varphi_g\right) + \left((\rho-1)\frac{1}{s}\frac{1}{\gamma_1} + \delta_0\right)\varphi_g\right)x_t + (\rho-1)\frac{1}{s}(1-s)\left(\frac{1}{\gamma_1}\varphi_g - 1\right)\eta[B]\,x_{t-1} \\
&+ \left(-(\rho-1)\frac{1}{s}(1-s)\frac{1}{\gamma_1}\eta[B] - \delta[B]\right)x_t + (\delta[B]-\delta_0)\,x_{t+1}.
\end{aligned} \tag{E10}$$

Using the identification approach, we deduce

$$\log(\zeta) = D_0 + \frac{1}{2}\left((\rho-1)(s\gamma_1)^{-1} + \delta_0\right)^2 \gamma_1^2 \upsilon, \tag{E11}$$

and

$$\begin{aligned}
\delta_1 &= -\left((\rho-1)\frac{1}{s} + \delta_0\varphi_g\right) - \left(-(\rho-1)\frac{1}{s}(1-s)\frac{1}{\gamma_1}\eta_0 - \delta_0\right), \\
\delta_{j+1} &= -(\rho-1)\frac{1}{s}(1-s)\left(\frac{1}{\gamma_1}\varphi_g - 1\right)\eta_{j-1} - \left(-(\rho-1)\frac{1}{s}(1-s)\frac{1}{\gamma_1}\eta_j - \delta_j\right) \text{ for } j \geq 1.
\end{aligned} \tag{E12}$$

Exploiting the solution to the eigenfunction function, we derive the transitory component of the SDF as

$$m^T_{t,t+1} = \exp\left(D_0 + D_1 + D_5\,(x_t - x_{t+1})\right). \tag{E13}$$

Equation (E13) implies the permanent component in equation (33) of Proposition 1, where

$$D_0 = \log(\beta) + (\rho-1)\log(g) + (\rho-1)\frac{(s-1)}{s}\left(\frac{\eta_0}{1-\varphi_h} - 1\right)\log(g), \quad \text{(E14)}$$

$$D_1 = \frac{1}{2}\left((\rho-1)(s\gamma_1)^{-1} + \delta_0\right)^2 \gamma_1^2 \upsilon, \quad \text{(E15)}$$

$$D_2 = (\rho-1)\frac{1}{s}(1-s)\left(\frac{1}{\gamma_1}\varphi_g - 1\right)\eta[B], \quad \text{(E16)}$$

$$D_3 = -\delta[B] - (\rho-1)\frac{1}{s}(1-s)\frac{1}{\gamma_1}\eta[B] + (\rho-1)\frac{1}{s}\left(1 - \frac{1}{\gamma_1}\varphi_g\right), \quad \text{(E17)}$$

$$D_4 = (\rho-1)(s\gamma_1)^{-1} + \delta[B], \quad \text{(E18)}$$

$$D_5 = \delta[B]. \quad \text{(E19)}$$

This ends the proof. ■

Online Appendix IV: Details of the recursive utility models in Propositions 2 and 3

Based on equations (34) and (36), we note that ω_{gt}, z_{gt}, and ω_{ht} are standard normal random variables, independent of each other and across time. The jump component z_{gt} is a Poisson mixture of normals: conditional on the number of jumps j, z_{gt} is normal with mean $j\theta$ and variance $j\delta^2$. The probability of $j \geq 0$ jumps at date $t+1$ is $e^{h_t} h_t^j / j!$ expands to

$$m_{t,t+1} = \exp\left(\chi_0 + a_g[B]\upsilon_t^{\frac{1}{2}}\omega_{gt+1} + a_z[B]z_{gt+1} + a_\upsilon[B]\omega_{\upsilon t+1} + a_h[B]\omega_{ht+1}\right), \quad \text{(F1)}$$

$$\chi_0 = \log(\beta) + (\rho-1)\log(g) - (\alpha-\rho)(D\upsilon - Jh) - (\alpha-\rho)(\alpha/2)\left((Db_1\nu[b_1])^2 + (Jb_1\eta[b_1])^2\right), \quad \text{(F2)}$$

where $a_g[B]$, $a_z[B]$, $a_\upsilon[B]$, and $a_h[B]$ are backshift operators defined as follows:

$$a_g[B] = (\rho-1)\gamma[B] + (\alpha-\rho)\gamma[b_1], \qquad a_z[B] = (\rho-1)\psi[B] + (\alpha-\rho)\psi[b_1], \quad \text{(F3)}$$

$$a_\upsilon[B] = (\alpha-\rho)D(b_1\nu[b_1] - \nu[B]B), \qquad a_h[B] = (\alpha-\rho)J(b_1\eta[b_1] - \eta[B]B), \quad \text{(F4)}$$

$$D = (\alpha/2)(\gamma[b_1])^2, \quad \text{and} \qquad J = \left(\frac{e^{\alpha\psi[b_1]\theta + (\alpha\psi[b_1]\delta)^2} - 1}{\alpha}\right). \quad \text{(F5)}$$

The functions $\eta[b_1]$, $\nu[b_1]$, and $\gamma[b_1]$ are polynomial functions of b_1:

$$\eta[b_1] = \sum_{j=0}^{\infty} b_1^j \eta_j, \quad \gamma[b_1] = \sum_{j=0}^{\infty} b_1^j \gamma_j, \quad \nu[b_1] = \sum_{j=0}^{\infty} b_1^j \nu_j, \quad \psi[b_1] = \sum_{j=0}^{\infty} b_1^j \psi_j, \quad \text{(F6)}$$

with $\gamma_0 = 1$ where,

$$\sum_{j=1}^{\infty} \gamma_j < \infty, \quad \sum_{j=1}^{\infty} \eta_j < \infty, \quad \sum_{j=1}^{\infty} \nu_j < \infty, \quad \sum_{j=1}^{\infty} \psi_j < \infty, \tag{F7}$$

and

$$\nu[B] = \sum_{j=0}^{\infty} \nu_j B^j \qquad \text{and} \qquad \psi[B] = \sum_{j=0}^{\infty} \psi_j B^j. \tag{F8}$$

A. Recursive utility with stochastic variance: The SDF is a special case of (F1) with $h = 0$, $\eta[B] = 0$, $J = 0$. The SDF takes the following form

$$m_{t,t+1} = \exp\left(\begin{array}{c} H_0 + (\rho - 1)\gamma[B]\upsilon_t^{\frac{1}{2}}\omega_{gt+1} + (\alpha - \rho)\gamma[b_1]\upsilon_t^{\frac{1}{2}}\omega_{gt+1} \\ +(\alpha - \rho)Db_1\nu[b_1]\omega_{\upsilon t+1} - (\alpha - \rho)D\nu[B]B\omega_{\upsilon t+1} \end{array} \right),$$

with

$$H_0 = \log(\beta) + (\rho - 1)\log g - (\alpha - \rho)(D\upsilon) - (\alpha - \rho)(\alpha/2)\left((Db_1\nu[b_1])^2\right). \tag{F9}$$

Now, define

$$x_t = (\gamma[B] - \gamma_0)\upsilon_t^{\frac{1}{2}}\omega_{gt+1}. \tag{F10}$$

The state variable x_t dynamics is:

$$x_t = \varphi_g x_{t-1} + \gamma_1 \upsilon_{t-1}^{\frac{1}{2}}\omega_{gt}, \qquad \text{with} \qquad \gamma_j = \varphi_g \gamma_{j-1} \text{ for } j \geq 2 \qquad \text{and} \qquad \varphi_g = \frac{\gamma_2}{\gamma_1}. \tag{F11}$$

It can also be shown that the dynamics of the state variable υ_t is:

$$\upsilon_t - \upsilon = \varphi_\upsilon(\upsilon_{t-1} - \upsilon) + \nu_0\omega_{\upsilon t}, \qquad \text{for } j \geq 2 \qquad \text{and} \qquad \varphi_\upsilon = \frac{\nu_1}{\nu_0}. \tag{F12}$$

The SDF can be expanded to

$$m_{t,t+1} = \exp(H_1 + H_2 x_t + H_3 x_{t+1} + H_4 \upsilon_t + H_5 \upsilon_{t+1}), \tag{F13}$$

where,

$$H_1 = H_0 + (\alpha-\rho)D\upsilon + (\alpha-\rho)Db_1\nu[b_1]\frac{(\varphi_\upsilon - 1)}{\nu_0}\upsilon, \tag{F14}$$

$$H_2 = (\rho-1) - ((\alpha-\rho)\gamma[b_1] + (\rho-1))\frac{\varphi_g}{\gamma_1}, \tag{F15}$$

$$H_3 = \frac{(\rho-1)}{\gamma_1} + \frac{(\alpha-\rho)\gamma[b_1]}{\gamma_1}, \tag{F16}$$

$$H_4 = (\alpha-\rho)D\left(-b_1\nu[b_1]\frac{\varphi_\upsilon}{\nu_0} - 1\right), \text{ and} \tag{F17}$$

$$H_5 = (\alpha-\rho)Db_1\frac{\nu[b_1]}{\nu_0}. \tag{F18}$$

Proceeding, we now solve the eigenfunction problem specified in equations (D1) and (D2). We conjecture that $\log(e_{t+1}) = \tau_0 x_{t+1} + \tau_1 \upsilon_{t+1}$. Hence,

$$\log(m_{t,t+1}e_{t+1}/e_t) = H_1 + (H_2-\tau_0)x_t + (H_3+\tau_0)x_{t+1} + (H_4-\tau_1)\upsilon_t + (H_5+\tau_1)\upsilon_{t+1}, \tag{F19}$$

and

$$\begin{aligned}\log(\zeta) &= H_1 + (H_5+\tau_1)\upsilon(1-\varphi_\upsilon) + \frac{1}{2}(H_5+\tau_1)^2\nu_0^2 + (H_2-\tau_0+(H_3+\tau_0)\varphi_g)x_t \\ &\quad + \left((H_4-\tau_1) + \frac{1}{2}(H_3+\tau_0)^2\gamma_1^2 + (H_5+\tau_1)\varphi_\upsilon\right)\upsilon_t. \end{aligned} \tag{F20}$$

Using the identification approach, we arrive at the expressions:

$$\log(\zeta) = H_1 + (H_5+\tau_1)\upsilon(1-\varphi_\upsilon) + \frac{1}{2}(H_5+\tau_1)^2\nu_0^2, \tag{F21}$$

and

$$\tau_0 = \frac{H_2 + H_3\varphi_g}{1-\varphi_g} \qquad \text{and} \qquad \tau_1 = \frac{H_4 + \frac{1}{2}(H_3+\tau_0)^2\gamma_1^2 + H_5\varphi_\upsilon}{1-\varphi_\upsilon}. \tag{F22}$$

With these results, we are in a position to state the transitory and permanent components as:

$$\begin{aligned} m^T_{t,t+1} &= \exp\left(H_1 + (H_5+\tau_1)\upsilon(1-\varphi_\upsilon) + \tfrac{1}{2}(H_5+\tau_1)^2\nu_0^2 + \tau_0(x_t - x_{t+1}) + \tau_1(\upsilon_t - \upsilon_{t+1}) \right), \\ m^P_{t,t+1} &= \exp\begin{pmatrix} -(H_5+\tau_1)\upsilon(1-\varphi_\upsilon) - \tfrac{1}{2}(H_5+\tau_1)^2\nu_0^2 \\ (H_2-\tau_0)x_t + (H_3+\tau_0)x_{t+1} + (H_4-\tau_1)\upsilon_t + (H_5+\tau_1)\upsilon_{t+1} \end{pmatrix}. \end{aligned} \tag{F23}$$

Setting $H_6 \equiv -(H_5+\tau_1)\upsilon(1-\varphi_\upsilon)-(H_5+\tau_1)^2\nu_0^2/2$, we obtain equation (38) of Proposition 2. ■

B. Recursive utility model with constant jump intensity: Consider the consumption growth dynamics with $\nu[B]=0$ (in this case $\upsilon_t=\upsilon$). It can be shown that the SDF reduces to

$$m_{t,t+1}=\exp\left(\begin{array}{c}\chi_0\\ +(\rho-1)x_t+((\rho-1)\gamma_0+(\alpha-\rho)\gamma[b_1])\,\upsilon^{\frac{1}{2}}\omega_{gt+1}\\ +(\rho-1)(\psi[B]-\psi_0)z_{gt+1}+((\rho-1)\psi_0+(\alpha-\rho)\psi[b_1])z_{gt+1}\\ +(\alpha-\rho)Jb_1\eta[b_1]\omega_{ht+1}-(\alpha-\rho)(h_t-h)J\end{array}\right). \tag{F24}$$

Now denote

$$\widetilde{x}_t=(\psi[B]-\psi_0)z_{gt+1}. \tag{F25}$$

The law of motion of $\widetilde{x}_t$ becomes

$$\widetilde{x}_t=\varphi_z\widetilde{x}_{t-1}+\psi_1 z_{gt} \qquad \text{with} \qquad \varphi_z=\frac{\psi_2}{\psi_1} \qquad \text{and} \qquad \psi_{j+2}=\varphi_z\psi_{j+1} \quad \text{for } j\geq 1. \tag{F26}$$

The SDF in equation (F24) reduces to

$$m_{t,t+1}=\exp\left(G_0+G_1x_t+G_2\widetilde{x}_{t-1}+G_3z_{gt}+G_4h_t+G_5z_{gt+1}+G_6\upsilon^{\frac{1}{2}}\omega_{gt+1}+G_7\omega_{ht+1}\right), \tag{F27}$$

with

$$\begin{aligned}
G_0 &= \chi_0+(\alpha-\rho)hJ, & G_1 &= (\rho-1),\\
G_2 &= (\rho-1)\varphi_z, & G_3 &= (\rho-1)\psi_1,\\
G_4 &= -(\alpha-\rho)J, & G_5 &= (\rho-1)\psi_0+(\alpha-\rho)\psi[b_1],\\
G_6 &= (\rho-1)\gamma_0+(\alpha-\rho)\gamma[b_1], & G_7 &= (\alpha-\rho)Jb_1\eta[b_1].
\end{aligned}$$

For the eigenfunction problem in equations (D1)-(D2), i.e., $E_t[m_{t,t+1}e_{t+1}]=\zeta e_t$, we conjecture that the eigenfunction is of the form:

$$e_{t+1}=\exp(\varsigma_0h_{t+1}+\varsigma_1z_{gt+1}+\varsigma_2x_{t+1}+\varsigma_3\widetilde{x}_t). \tag{F28}$$

Algebraic manipulation yields the expression:

$$
\begin{aligned}
m_{t,t+1}\frac{e_{t+1}}{e_t} &= \exp\left(\begin{array}{c} G_0+\varsigma_0 h-\varsigma_0\varphi_h h+(G_1-\varsigma_2+\varsigma_2\varphi_g)\,x_t+G_2\widetilde{x}_{t-1}+(G_3-\varsigma_1+\varsigma_3\psi_1)\,z_{gt} \\ +(G_4-\varsigma_0+\varsigma_0\varphi_h)\,h_t+(\varsigma_3\varphi_z\ -\varsigma_3)\,\widetilde{x}_{t-1} \end{array}\right) \\
&\quad\times\exp\left((G_5+\varsigma_1)\,z_{gt+1}+(G_6+\varsigma_2\gamma_1)\,\upsilon^{\frac{1}{2}}\omega_{gt+1}+(G_7+\varsigma_0\eta_0)\,\omega_{ht+1}\right). \qquad \text{(F29)}
\end{aligned}
$$

Upon further manipulation of equation (F29), we get

$$
\zeta = \xi\times\exp\left(\begin{array}{c} G_0+\varsigma_0 h-\varsigma_0\varphi_h h+(G_1-\varsigma_2+\varsigma_2\varphi_g)\,x_t+G_2\widetilde{x}_{t-1}+(G_3-\varsigma_1+\varsigma_3\psi_1)\,z_{gt} \\ +(G_4-\varsigma_0+\varsigma_0\varphi_h)\,h_t+(\varsigma_3\varphi_z\ -\varsigma_3)\,\widetilde{x}_{t-1} \end{array}\right), \qquad \text{(F30)}
$$

with

$$
\xi=E_t\left(\exp\left((G_5+\varsigma_1)\,z_{gt+1}+(G_6+\varsigma_2\gamma_1)\,\upsilon^{\frac{1}{2}}\omega_{gt+1}+(G_7+\varsigma_0\eta_0)\,\omega_{ht+1}\right)\right). \qquad \text{(F31)}
$$

One may observe that

$$
\begin{aligned}
\xi &= \left(E_t\exp\left((G_5+\varsigma_1)\,z_{gt+1}\right)\right)\left(E_t\exp\left((G_6+\varsigma_2\gamma_1)\,\upsilon^{\frac{1}{2}}\omega_{gt+1}\right)\right)\left(E_t\left((G_7+\varsigma_0\eta_0)\,\omega_{ht+1}\right)\right) \qquad \text{(F32)} \\
&= E_t\left(\exp\left(\left((G_5+\varsigma_1)\,\theta+\frac{1}{2}(G_5+\varsigma_1)^2\delta^2\right)j\right)\right)\exp\left(\frac{1}{2}(G_6+\varsigma_2\gamma_1)^2\upsilon+\frac{1}{2}(G_7+\varsigma_0\eta_0)^2\right),
\end{aligned}
$$

and

$$
E_t\left(\exp\left(\left((G_5+\varsigma_1)\,\theta+\frac{1}{2}(G_5+\varsigma_1)^2\delta^2\right)j\right)\right)=\exp\left(G_8 h_t\right), \qquad \text{(F33)}
$$

with

$$
G_8=e^{\left((G_5+\varsigma_1)\theta+\frac{1}{2}(G_5+\varsigma_1)^2\delta^2\right)}-1. \qquad \text{(F34)}
$$

As a consequence, equation (F32) simplifies to

$$
\xi = \exp\left(G_8 h_t+\frac{1}{2}(G_6+\varsigma_2\gamma_1)^2\upsilon+\frac{1}{2}(G_7+\varsigma_0\eta_0)^2\right). \qquad \text{(F35)}
$$

We substitute equation (F35) equation in equation (F30) and rearrange to obtain:

$$
\begin{aligned}
\log(\varsigma) &= G_0+\varsigma_0 h-\varsigma_0\varphi_h h+(G_1-\varsigma_2+\varsigma_2\varphi_g)\,x_t \\
&\quad+(G_3-\varsigma_1+\varsigma_3\psi_1)\,z_{gt}+(G_4-\varsigma_0+\varsigma_0\varphi_h+G_8)\,h_t \\
&\quad+(\varsigma_3\varphi_z\ -\varsigma_3+G_2)\,\widetilde{x}_{t-1}+\frac{1}{2}(G_6+\varsigma_2\gamma_1)^2\upsilon+\frac{1}{2}(G_7+\varsigma_0\eta_0)^2. \qquad \text{(F36)}
\end{aligned}
$$

Using the identification approach, we then have

$$\log(\varsigma) = G_0 + \varsigma_0 h\left(1-\varphi_h\right) + \frac{1}{2}\left(G_6 + \varsigma_2\gamma_1\right)^2 \upsilon + \frac{1}{2}\left(G_7 + \varsigma_0\eta_0\right)^2, \tag{F37}$$

and

$$\begin{aligned} G_1 - \varsigma_2 + \varsigma_2\varphi_g &= 0, & G_4 - \varsigma_0 + \varsigma_0\varphi_h + G_8 &= 0, \\ G_3 - \varsigma_1 + \varsigma_3\psi_1 &= 0, & \varsigma_3\varphi_z - \varsigma_3 + G_2 &= 0. \end{aligned} \tag{F38}$$

Finally, we get

$$\varsigma_0 = \tfrac{G_8+G_4}{1-\varphi_h}, \quad \varsigma_1 = G_3 + \varsigma_3\psi_1, \quad \varsigma_2 = \tfrac{G_1}{1-\varphi_g}, \quad \varsigma_3 = \tfrac{G_2}{1-\varphi_z}. \tag{F39}$$

The transitory component is, therefore, $m^T_{t,t+1} = \zeta \exp\left(e_t - e_{t+1}\right)$, and we obtain:

$$m^T_{t,t+1} = \zeta \exp\left(\varsigma_0\left(h_t - h_{t+1}\right) + \varsigma_1\left(z_{gt} - z_{gt+1}\right) + \varsigma_2\left(x_t - x_{t+1}\right) + \varsigma_3\left(\widetilde{x}_{t-1} - \widetilde{x}_t\right)\right). \tag{F40}$$

We can establish the relation in equation (39) of Proposition 3 by setting $G_9 \equiv -\frac{1}{2}\left(G_6 + \varsigma_2\gamma_1\right)^2 \upsilon - \frac{1}{2}\left(G_7 + \varsigma_0\eta_0\right)^2$. ■

Online Appendix V: Proof of Theorem 3

Proof of the n-period bound for the permanent component of the SDF: The entropy is

$$L\left[m^P_{t,t+n}\right] = \log\left(E\left[m^P_{t,t+n}\right]\right) - E\left[\log\left(m^P_{t,t+n}\right)\right]. \tag{G1}$$

Using Jensen's inequality, we have

$$\begin{aligned} E\left[\log\left(m_{t,t+n}\mathbf{w}^{(n)'}\mathbf{R}_{t,t+n}\right)\right] &= E\left[\log\left(m^P_{t,t+n}\mathbf{w}^{(n)'}\mathbf{R}_{t,t+n}/R_{t,t+n,\infty}\right)\right], \\ &\leq \log\left(E\left[m_{t,t+n}\mathbf{w}^{(n)'}\mathbf{R}_{t,t+n}\right]\right), \\ &\leq \log\left(\mathbf{w}^{(n)'}\mathbf{1}_n\right) = \log\left(1\right), \\ &\leq 0, \end{aligned} \tag{G2}$$

and

$$E\left[\log\left(\mathbf{w}^{(n)'}\mathbf{R}_{t,t+n}\right)\right] - E\left[\log\left(R_{t,t+n,\infty}\right)\right] \leq -E\left[\log\left(m^P_{t,t+n}\right)\right]. \tag{G3}$$

Adding $\log(1) = \log\left(E\left[m^P_{t,t+n}\right]\right)$ to both sides of equation (G3) yields

$$E\left[\log\left(\mathbf{w}^{(n)'}\mathbf{R}_{t,t+n}\right)\right] - E\left[\log\left(R_{t,t+n,\infty}\right)\right] \leq \log\left(E\left[m^P_{t,t+n}\right]\right) - E\left[\log\left(m^P_{t,t+n}\right)\right] = L\left[m^P_{t,t+n}\right], \tag{G4}$$

and

$$\begin{aligned}
L\left[m^P_{t,t+n}\right] &\geq E\left[\log\left(\mathbf{w}^{(n)'}\mathbf{R}_{t,t+n}\right)\right] - E\left[\log\left(R_{t,t+n,\infty}\right)\right], \\
&\geq E\left[\log\left(\mathbf{w}^{(n)'}\mathbf{R}_{t,t+n}\right)\right] + \log\left(E\left[q^{(1)}_t\right]\right) - \log\left(E\left[q^{(1)}_t\right]\right) - E\log\left(R_{t,t+n,\infty}\right), \\
&\geq E\left[\log\left(\mathbf{w}^{(n)'}\mathbf{R}_{t,t+n}\right)\right] - n\log\left(1/E\left[q^{(1)}_t\right]\right) + n\log\left(1/E\left[q^{(1)}_t\right]\right) - E\left[\log\left(1/R_{t,t+n,\infty}\right)\right], \\
&\geq n\left(\mathrm{er}^{(n)}_R - \mathrm{er}^{(n)}_\infty\right),
\end{aligned} \tag{G5}$$

with $\mathrm{er}^{(n)}_\infty = \frac{1}{n}E\left[\log\left(R_{t,t+n,\infty}\right)\right] - \log\left(1/E\left[q^{(1)}_t\right]\right)$.

The entropy of $\left(m^P_{t,t+n}\right)^2$ is

$$L[\left(m^P_{t,t+n}\right)^2] = \log(E\left[(m^P_{t,t+n})^2\right]) + 2L[m^P_{t,t+n}]. \tag{G6}$$

By virtue of equation (G5), $L\left[m^P_{t,t+n}\right] \geq n\left(\mathrm{er}^{(n)}_R - \mathrm{er}^{(n)}_\infty\right)$, equation (G6) reduces to

$$L[\left(m^P_{t,t+n}\right)^2] \geq \log(E[(m^P_{t,t+n})^2]) + 2n\left(\mathrm{er}^{(n)}_R - \mathrm{er}^{(n)}_\infty\right). \tag{G7}$$

Using the Cauchy Schwartz inequality, we deduce

$$Var\left[m^P_{t,t+n}\right] \geq \mathbf{y}^{(n)'}_P \Sigma^{(n)}_P \mathbf{y}^{(n)}_P. \tag{G8}$$

Hence,

$$L\left[\left(m^P_{t,t+n}\right)^2\right] \geq \mathrm{v}^{(n)}_P + 2n\left(\mathrm{er}^{(n)}_R - \mathrm{er}^{(n)}_\infty\right) \qquad \text{with} \qquad \mathrm{v}^{(n)}_P \equiv \log\left(1 + \mathbf{y}^{(n)'}_P \Sigma^{(n)}_P \mathbf{y}^{(n)}_P\right). \tag{G9}$$

This ends the proof. ■

Proof of the n-period bounds for the SDF: We have $L\left[m_{t,t+n}\right] = \log\left(E\left[m_{t,t+n}\right]\right) - E\left[\log\left(m_{t,t+n}\right)\right]$ and

$q_t^{(n)} = E_t\left[m_{t,t+n}\right]$. Using Jensen's inequality, we have

$$\begin{aligned}
E\left[\log\left(m_{t,t+n}\mathbf{w}^{(n)'}\mathbf{R}_{t,t+n}\right)\right] &\leq \log\left(E\left[m_{t,t+n}\mathbf{w}^{(n)'}\mathbf{R}_{t,t+n}\right]\right), \\
&\leq \log\left(\mathbf{w}^{(n)'}E\left[m_{t,t+n}\mathbf{R}_{t,t+n}\right]\right), \\
&\leq \log\left(\mathbf{w}^{(n)'}\mathbf{1}\right), \\
&\leq 0, \qquad \text{(G10)}
\end{aligned}$$

and

$$E\left[\log\left(\mathbf{w}^{(n)'}\mathbf{R}_{t,t+n}\right)\right] \leq -E\left[\log\left(m_{t,t+n}\right)\right]. \qquad \text{(G11)}$$

Adding $\log\left(E\left[m_{t,t+n}\right]\right)$ to both sides of equation (G3) yields

$$\log\left(E\left[m_{t,t+n}\right]\right) + E\left[\log\left(\mathbf{w}^{(n)'}\mathbf{R}_{t,t+n}\right)\right] \leq \log\left(E\left[m_{t,t+n}\right]\right) - E\left[\log\left(m_{t,t+n}\right)\right] = L\left[m_{t,t+n}\right], \qquad \text{(G12)}$$

and

$$\begin{aligned}
L\left[m_{t,t+n}\right] &\geq E\left[\log\left(\mathbf{w}^{(n)'}\mathbf{R}_{t,t+n}\right)\right] + \log\left(E\left[q_t^{(n)}\right]\right), \\
&\geq E\left[\log\left(\mathbf{w}^{(n)'}\mathbf{R}_{t,t+n}\right)\right] - \log\left(E\left[q_t^{(1)}\right]\right) + \log\left(E\left[q_t^{(1)}\right]\right) + \log\left(E\left[q_t^{(n)}\right]\right), \\
&\geq E\left[\log\left(\mathbf{w}^{(n)'}\mathbf{R}_{t,t+n}\right)\right] + \log\left(E\left[q_t^{(1)}\right]\right) - \log\left(E\left[q_t^{(1)}\right]\right) + \log\left(E\left[q_t^{(n)}\right]\right), \\
&\geq E\left[\log\left(\mathbf{w}^{(n)'}\mathbf{R}_{t,t+n}\right)\right] - n\log\left(1/E\left[q_t^{(1)}\right]\right) - \left(n\log\left(E\left[q_t^{(1)}\right]\right) - \log\left(E\left[q_t^{(n)}\right]\right)\right), \\
&\geq n\left(\mathrm{er}_R^{(n)} - \mathrm{Q}^{(n)}\right), \qquad \text{(G13)}
\end{aligned}$$

with

$$\mathrm{er}_R^{(n)} = E\left[\log\left(\left(\mathbf{w}^{(n)'}\mathbf{R}_{t,t+n}\right)^{1/n}\right)\right] - \log\left(1/E\left[q_t^{(1)}\right]\right) \text{ and } \mathrm{Q}^{(n)} = \log\left(E\left[q_t^{(1)}\right]\right) - \frac{1}{n}\log\left(E\left[q_t^{(n)}\right]\right). \qquad \text{(G14)}$$

Now note the relation $L\left[m_{t,t+n}^2\right] = \log\left(E\left[m_{t,t+n}^2\right]\right) - E\left[\log\left(m_{t,t+n}^2\right)\right]$. We expand $L\left[m_{t,t+n}^2\right]$ to

$$\begin{aligned}
L\left[m_{t,t+n}^2\right] &= \log\left(E\left[m_{t,t+n}^2\right]\right) - 2E\left[\log\left(m_{t,t+n}\right)\right], \\
&= \log\left(E\left[m_{t,t+n}^2\right]\right) - 2\log\left(E\left[m_{t,t+n}\right]\right) + 2\left(\log\left(E\left[m_{t,t+n}\right]\right) - E\left[\log\left(m_{t,t+n}\right)\right]\right), \\
&= \log\left(E\left[m_{t,t+n}^2\right]\right) - 2\log\left(E\left[q_t^{(n)}\right]\right) + 2L\left[m_{t,t+1}\right]. \qquad \text{(G15)}
\end{aligned}$$

Recall that

$$
\begin{aligned}
L[m_{t,t+n}^2] &= \log\left(E\left[m_{t,t+n}^2\right]\right) - 2E\left[\log\left(m_{t,t+n}\right)\right], \\
&= \log\left(E\left[m_{t,t+n}^2\right]\right) - 2\log\left(E\left[q_t^{(n)}\right]\right) + 2L\left[m_{t,t+n}\right], \\
&= \log\left(\frac{E\left[m_{t,t+n}^2\right]}{\left(E\left[q_t^{(n)}\right]\right)^2}\right) + 2L\left[m_{t,t+n}\right].
\end{aligned} \tag{G16}
$$

Invoking equation (G13), that is, $L\left[m_{t,t+n}\right] \geq n\left(\mathrm{er}_R^{(n)} - \mathrm{Q}^{(n)}\right)$, equation (G16) reduces to

$$
L\left[m_{t,t+n}^2\right] \;\geq\; \log\left(\frac{E\left[m_{t,t+n}^2\right]}{\left(E\left[q_t^{(n)}\right]\right)^2}\right) \;+\; 2n\left(\mathrm{er}_R^{(n)} - \mathrm{Q}^{(n)}\right). \tag{G17}
$$

We notice that

$$
\log\left(\frac{E\left[m_{t,t+n}^2\right]}{\left(E\left[q_t^{(n)}\right]\right)^2}\right) = \log\left(1 + \frac{E\left[m_{t,t+n}^2\right] - \left(E\left[q_t^{(n)}\right]\right)^2}{\left(E\left[q_t^{(n)}\right]\right)^2}\right). \tag{G18}
$$

Using the Cauchy Schwartz inequality, it can be shown that

$$
Var\left[m_{t,t+n}\right] \;\geq\; \left(\mathbf{1} - \left(E\left[q_t^{(n)}\right]\right)E\left[\mathbf{R}_{t,t+n}\right]\right)' \,\Sigma^{(n)}\, \left(\mathbf{1} - \left(E\left[q_t^{(n)}\right]\right)E\left[\mathbf{R}_{t,t+n}\right]\right). \tag{G19}
$$

This result implies that

$$
L\left[m_{t,t+n}^2\right] \geq \mathrm{v}_R^{(n)} + 2n\left(\mathrm{er}_R^{(n)} - \mathrm{Q}^{(n)}\right) \quad \text{with} \quad \mathrm{v}_P^{(n)} \equiv \log\left(1 + \mathbf{y}^{(n)\prime}\,\Sigma^{(n)}\,\mathbf{y}^{(n)} / \left(E[q_t^{(n)}]\right)^2\right). \tag{G20}
$$

We consequently have the complete expression. ■

Table 1

Sharpness of our entropy bounds on $m^P_{t,t+1}$ and $m_{t,t+1}$

The Alvarez and Jermann (2005, equation (4)) lower bound on the entropy of $m^P_{t,t+1}$ (denoted by AJ) is based on the expression:

$$\text{AJ}: \quad E[\log(R^m_{t,t+1})] - E[\log(R_{t,t+1,\infty})],$$

while the Backus, Chernov, and Zin (2013, equation (5)) lower bound on the entropy of $m_{t,t+1}$ (denoted by BCZ) is based on the expression:

$$\text{BCZ}: \quad E[\log(R^m_{t,t+1})] - E[\log(R^f_t)],$$

where $R^m_{t,t+1}$ is the return on a single risky asset, which we proxy by the value-weighted equity market return. Moreover, $R_{t,t+1,\infty}$ is the return on an infinite-maturity bond, which we proxy by the return of a 30-year Treasury bond. R^f_t is the gross return of the three-month Treasury bond. Our lower entropy bounds on $m^P_{t,t+1}$ and $m_{t,t+1}$ are based on equations (21) and (22) of Theorem 1 and rely on ability of the SDF to correctly price $N+2$ assets (the risk-free bond, the long-term bond, and N risky assets). The N risky assets are based on two data sets: SET A contains the value-weighted market returns together with the 25 Fama-French size and book-to-market portfolios, while SET B contains the value-weighted market returns together with the 25 Fama-French size and momentum portfolios. The sample period is from July 1931 to December 2011 (966 observations). Reported are the lower entropy bounds, with the one-sided p-values in $\langle . \rangle$. To compute these p-values, we first use the block bootstrap with a block size of 20 to generate 50,000 samples from the original data. Then we compute the lower bounds in each sample and tabulate the proportion of bootstrap samples for which the lower bound is less than zero.

	AJ	BCZ	Our entropy bounds	
	(Eq. (4))	(Eq. (5))	Market + 25 size & B/M (SET A)	Market + 25 size & momentum (SET B)
$L[m^P_{t,t+1}]$	0.0030		0.0214	0.0348
	$\langle 0.067 \rangle$		$\langle 0.000 \rangle$	$\langle 0.003 \rangle$
$L[m_{t,t+1}]$		0.0050	0.0233	0.0367
		$\langle 0.003 \rangle$	$\langle 0.000 \rangle$	$\langle 0.003 \rangle$

Table 2

Relevance of our entropy bounds on $(m^P_{t,t+1})^2$

The logic of this test is that when the permanent component of the SDF is lognormally distributed with no time-variation in the conditional volatility (mean) of the log permanent component of the SDF, then $L[(m^P_{t,t+1})^2] = 4\,L[m^P_{t,t+1}]$. Guided by Theorem 1, the ratio of the lower bound on $L[(m^P_{t,t+1})^2]$ to four times the lower bound on $L[m^P_{t,t+1}]$ is equal to 1. Define

$$\Pi \;=\; \frac{2\,(\mathrm{er}_R \;-\; \mathrm{er}_\infty) \;+\; \mathrm{v}_P}{4(\mathrm{er}_R \;-\; \mathrm{er}_\infty)} \;-\; 1,$$

with

$$\begin{array}{lll} \mathrm{er}_R \equiv E[\log(\mathbf{w}'\mathbf{R}_{t,t+1})] \;-\; \log\left((E[q_t])^{-1}\right) & \text{and} & \mathrm{er}_\infty \equiv E[\log(R_{t,t+1,\infty})] \;-\; \log((E[q_t])^{-1}), \\ \mathrm{v}_P \equiv \log\left(1+\mathbf{y}_P'\Sigma_P\,\mathbf{y}_P\right) & \text{and} & \mathbf{y}_P \equiv \Sigma_P^{-1}\left(\mathbf{1}-E\left[\mathbf{R}_{t,t+1}/R_{t,t+1,\infty}\right]\right), \\ \mathbf{w} \equiv \mathbf{y}/\left(\mathbf{1}'\mathbf{y}\right) & \text{and} & \mathbf{y} \equiv \Sigma^{-1}\left(\mathbf{1}-E\left[q_t\right]E\left[\mathbf{R}_{t,t+1}\right]\right), \end{array}$$

where $\mathbf{R}_{t,t+1}$ is a set of risky asset returns and $R_{t,t+1,\infty}$ is the return on an infinite-maturity discount bond. In our implementation, we proxy $R_{t,t+1,\infty}$ by the monthly return of a 30-year Treasury bond. Σ_P is the variance co-variance matrix of $\mathbf{R}_{t,t+1}/R_{t,t+1,\infty}$. Σ is the variance covariance matrix of $\mathbf{R}_{t,t+1}$.

Our computation of Π relies on three data sets for $\mathbf{R}_{t,t+1}$: SET A contains the value-weighted market returns together with the 25 Fama-French size and book-to-market portfolios; SET B contains the value-weighted market returns together with the 25 Fama-French size and momentum portfolios; while SET C contains only the value-weighted equity market returns. The sample period is from July 1931 to December 2011 (966 observations). To compute the p-values reported in parentheses, we employ a block bootstrap with a block size of 20 to generate $\widehat{b}$=50,000 samples from the original data. We then compute $\Pi_b = \Pi$ for $b = 1,\ldots,\widehat{b}$, the cross-sectional average $\overline{\Pi}$, and the standard error se(Π) =std$(\Pi)\,/\sqrt{\widehat{b}}$ of Π. Accordingly, we compute the t statistic as $\left(\overline{\Pi}-0\right)/se\,(\Pi)$. The absolute value of the t-statistic is then used to compute the two-sided p-value.

	Testing H_0: $\Pi = 0$ versus H_a: $\Pi \neq 0$		
	Market+25 size & B/M (SET A)	Market+25 size & momentum (SET B)	Market only (SET C)
$\Pi = \frac{2(\mathrm{er}_R \,-\, \mathrm{er}_\infty) \,+\, \mathrm{v}_P}{4(\mathrm{er}_R \,-\, \mathrm{er}_\infty)} - 1$	51.85%	33.05%	13.79%
	(0.000)	(0.000)	(0.021)

Table 3

Model comparisons based on the lower entropy bounds

Reported are the entropies of $(m^P_{t,t+1})^2$ and $m^P_{t,t+1}$ for difference habit (denoted by DH), the recursive utility with stochastic variance (denoted by RU-SV), and the recursive utility with constant jump intensity (denoted by RU-CJI). The one-sided p-values shown in square brackets represent the proportion of replications for which the model-based entropy exceeds, in 50,000 replications, the lower bound on the entropy computed from observed asset prices. Our lower entropy bounds on $m^P_{t,t+1}$ and $m_{t,t+1}$ are based on equations (21) and (22) of Theorem 1 and rely on the ability of the SDF to correctly price $N+2$ assets (the risk-free bond, the long-term discount bond, and N risky assets). The N risky assets are based on SET B, which contains the value-weighted market returns together with the 25 Fama-French size and momentum portfolios. The sample period is from July 1931 to December 2011. The lower entropy bounds on $(m_{t,t+1})^2$ and $m_{t,t+1}$ are analogously obtained based on Theorem 1. We focus on SET B, as it corresponds to the maximum lower bound on entropy measures (as in our Table 1). Panels C and D present the variance, skewness, and kurtosis of $m^P_{t,t+1}$ and $m_{t,t+1}$ that are consistent with model parameterizations in Table Appendix-I. The one-sided p-values $\langle . \rangle$ reported below the lower entropy bounds, represent the proportion of bootstrap samples for which the lower bound is less than zero.

	Habit model	Recursive utility models		
	DH	RU-SV	RU-CJI	Lower entropy bound (Set B)
Panel A: Entropies of $(m^P_{t,t+1})^2$ *and* $m^P_{t,t+1}$				
$L[(m^P_{t,t+1})^2]$	0.0811	0.095	1.4858	0.1851
	[0.000]	[0.000]	[1.000]	$\langle 0.003 \rangle$
$L[m^P_{t,t+1}]$	0.0203	0.0237	0.0197	0.0348
	[0.000]	[0.000]	[0.000]	$\langle 0.003 \rangle$
Panel B: Entropies of $(m_{t,t+1})^2$ *and* $m_{t,t+1}$				
$L[(m_{t,t+1})^2]$	0.0785	0.0869	1.4331	0.1956
	[0.000]	[0.000]	[1.000]	$\langle 0.003 \rangle$
$L[m_{t,t+1}]$	0.0196	0.0217	0.0190	0.0367
	[0.000]	[0.000]	[0.000]	$\langle 0.003 \rangle$
Panel C: Moments of the $m^P_{t,t+1}$ *distribution*				
Variance	0.0415	0.0487	3.2480	
Skewness	0.6142	0.6778	$+\infty$	
Kurtosis	3.6654	3.8786	$+\infty$	
Panel D: Moments of the $m_{t,t+1}$ *distribution*				
Variance	0.0403	0.0444	3.3438	
Skewness	0.6041	0.6476	$+\infty$	
Kurtosis	3.6447	3.8061	$+\infty$	

Table 4

Entropy-based measures of the transitory component of the SDF

Reported are the entropies of $(m^T_{t,t+1})^2$ and $m^P_{t,t+1}$ for three asset pricing models: the difference habit (denoted by DH), the recursive utility with stochastic variance (denoted by RU-SV), and the recursive utility with constant jump intensity (denoted by RU-CJI). The data-based $L[(m^T_{t,t+1})^2]$ and $L[m^T_{t,t+1}]$ rely on the expressions in equations (25) and (26), whereby we proxy $R_{t,t+1,\infty}$ by the return of a 30-year Treasury bond. The two-sided bootstrap p-values, shown in curly brackets, allow to test whether the average value of the model-implied entropy across the 50,000 replications is equal to the entropy-based measures computed from bond returns. Panel B presents the mean and standard deviation of the returns of the risk-free bond and the long-term implied by each model. Our replications are consistent with model parameterizations in Table Appendix-I.

	Model-implied entropies			
	DH	RU-SV	RU-CJI	Data implied
Panel A: Transitory component of the SDF				
$L[(m^T_{t,t+1})^2]$	2.8×10^{-3}	0.2×10^{-3}	0.046×10^{-3}	4.8×10^{-3}
	{0.000}	{0.000}	{0.000}	⟨0.000⟩
$L[m^T_{t,t+1}]$	0.7×10^{-3}	0.1×10^{-3}	0.012×10^{-3}	0.4×10^{-3}
	{0.000}	{0.000}	{0.000}	⟨0.000⟩
Panel B: Returns of the risk-free and the long-term discount bonds				
Mean of risk-free return	-0.0304	0.0112	-0.0160	0.0355
Std. Dev. of risk-free return	0.0342	0.0030	0.0006	0.0311
Mean of long-term bond return	-0.0225	-0.0124	-0.0153	0.0584
Std. Dev. of long-term bond return	0.4446	0.1323	0.0006	0.0355

Table 5

Entropy-based measures of codependence

Reported are the entropy-based codependence measures for three asset pricing models: the difference habit (denoted by DH), the recursive utility with stochastic variance (denoted by RU-SV), and the recursive utility with constant jump intensity (denoted by RU-CJI). The data-based $L[m^P_{t,t+1}m^T_{t,t+1}] - L[m^P_{t,t+1}] - L[m^T_{t,t+1}]$ is inferred from the Treasury yield curve, as described in equation (27). The p-values shown in curly brackets allow to test whether the average entropy-based codependence across the 50,000 values is equal to its data counterparts. The data-based $L[(m^P_{t,t+1}m^T_{t,t+1})^2] - L[(m^P_{t,t+1})^2] - L[(m^T_{t,t+1})^2]$ employs the expression on the right-hand side of equation (28) of Theorem 2. The construction of the upper bound relies on the risk-free bond, the long-term discount bond, along with Set A. Specifically, Set A contains the value-weighted market returns together with the 25 Fama-French size and book-to-market portfolios. The reported one-sided p-values, shown as $\lfloor . \rfloor$, represent the proportion of replications for which the model entropy-based bo-dependence do not exceed, in 50,000 replications, the upper bound on the codependence computed from asset returns. Our replications are consistent with model parameterizations in Table Appendix-I.

	Model-implied entropy codependence				
	DH	RU-SV	RU-CJI	Data implied	Upper bound (Set A)
$L[m^P_{t,t+1}m^T_{t,t+1}] - L[m^P_{t,t+1}] - L[m^T_{t,t+1}]$	-0.0014	-0.0021	-0.0007	0.0015	
	$\{0.000\}$	$\{0.000\}$	$\{0.000\}$	$\langle 0.031 \rangle$	
$L[(m^P_{t,t+1}m^T_{t,t+1})^2] - L[(m^P_{t,t+1})^2] - L[(m^T_{t,t+1})^2]$	-0.0054	-0.0083	-0.0028		0.1222
	$\lfloor 0.002 \rfloor$	$\lfloor 0.000 \rfloor$	$\lfloor 0.000 \rfloor$		$\langle 0.000 \rangle$

Table Appendix-I

Parameters employed in model implementation

Displayed in this table are the parameters that govern preferences and the dynamics of consumption growth. These parameters are adopted from Tables 2, 3, and 4 of Backus, Chernov, and Zin (2013), and likewise $\log(g)$ and η_0 are taken from their page 16. Our implementation of the models with difference habit (hereby, DH), recursive utility with stochastic variance (hereby, RU-SV), and recursive utility with constant jump intensity (RU-CJI) follows Backus, Chernov, and Zin (2013, respectively, Model (4) in Table 2, Model (1) in Table 3, and Model (4) in Table 4). We use US annual real personal consumption expenditures as a proxy for aggregate consumption over the sample period of 1931:07 to 2011:12 (966 observations). To compare model implications with the data, we simulate a finite sample of consumption growth, c_{t+1}/c_t, over 966 months. Following convention, we then compute the annualized consumption growth as $\exp(\sum_{j=1}^{12}\log(c_{t+j}/c_{t+j-1}))$. The reported model mean, standard deviation, and autocorrelation are based on the annualized consumption growth.

Parameter	DH	RU-SV	RU-CJI	Data implied 1931:07 to 2011:12
Panel A: Preferences				
ρ	-9.0000	0.3333	0.3333	
α		-9.0000	-9.0000	
β	0.9980	0.9980	0.9980	
φ_h	0.9000			
s	0.5000			
Panel B: Consumption growth dynamics				
γ_0	1.0000	1.0000	1.0000	
$\log(g)$	0.0015	0.0015	0.0015	
η_0	0.1000			
γ_1	0.0271	0.0271	0.0281	
φ_g	0.9790	0.9790	0.9690	
$\upsilon^{1/2}$	0.0099	0.0099	0.0079	
ν_0		0.23×10^{-5}		
φ_υ		0.9870		
h			0.0008	
θ			-0.1500	
δ			0.1500	
ψ_0			1.0000	
b_1		0.9977	0.9979	
Panel C: Consumption growth				
Mean (annualized)	1.0192	1.0190	1.0189	1.0339
Std. Dev. (annualized)	0.0416	0.0415	0.0369	0.0287
Autocorrelation	0.2424	0.2433	0.1771	0.2386

Table Appendix-II

Impact of alternative jump parameterizations in the RU-CJI model

Here we vary θ, δ, and h that govern the distribution of jumps (see equation (36)) in the consumption growth dynamics for the RU-CJI model. We keep other parameters of the RU-CJI model to those specified in Table Appendix-I. For each set of parameters, the reported values are averages across 50,000 replications. For each replication, we simulate the path of consumption growth c_{t+1}/c_t over 966 months. Following convention, we then compute the annualized consumption growth as $\exp(\sum_{j=1}^{12} \log(c_{t+j}/c_{t+j-1}))$. The reported model mean and standard deviation are based on the annualized consumption growth. For each parameter set, we also report the average values of entropy $L[m^P_{t,t+1}]$ and $L[(m^P_{t,t+1})^2]$, and the central moments of the permanent component of the SDF. The bolded parameter set corresponds to Backus, Chernov, and Zin (2013, Model (4), Table 4).

θ	δ	h	Entropies		Moments of $m^P_{t,t+1}$			$\frac{c_{t+1}}{c_t}$	
			$L[m^P]$	$L[(m^P)^2]$	Variance	Skewness	Kurtosis	Mean	Std. Dev.
-0.15	0.02	0.0002	0.011	0.046	0.024	3.62E+00	1.10E+02	1.0190	0.033
-0.15	0.02	0.0004	0.011	0.050	0.027	5.92E+00	1.93E+02	1.0187	0.034
-0.15	0.02	0.0008	0.012	0.057	0.033	8.95E+00	3.20E+02	1.0188	0.035
-0.15	0.07	0.0002	0.011	0.053	0.030	3.03E+01	6.96E+04	1.0187	0.033
-0.15	0.07	0.0004	0.012	0.062	0.039	4.51E+01	2.49E+06	1.0187	0.034
-0.15	0.07	0.0008	0.013	0.082	0.057	6.20E+01	3.95E+09	1.0188	0.036
-0.15	0.15	0.0002	0.013	0.403	0.459	4.82E+195	+∞	1.0187	0.034
-0.15	0.15	0.0004	0.015	0.764	1.083	+∞	+∞	1.0187	0.035
-0.15	**0.15**	**0.0008**	**0.020**	**1.486**	**3.248**	**+∞**	**+∞**	**1.0189**	**0.037**
-0.07	0.02	0.0002	0.011	0.043	0.022	5.63E-01	4.86E+00	1.0187	0.033
-0.07	0.02	0.0004	0.011	0.043	0.022	6.78E-01	6.30E+00	1.0187	0.033
-0.07	0.02	0.0008	0.011	0.044	0.023	8.98E-01	9.05E+00	1.0187	0.033
-0.07	0.07	0.0002	0.011	0.044	0.023	3.46E+00	2.94E+02	1.0186	0.033
-0.07	0.07	0.0004	0.011	0.046	0.024	6.03E+00	5.81E+02	1.0187	0.033
-0.07	0.07	0.0008	0.011	0.049	0.027	1.01E+01	1.17E+03	1.0187	0.034
-0.07	0.15	0.0002	0.012	0.115	0.096	1.79E+19	+∞	1.0187	0.033
-0.07	0.15	0.0004	0.012	0.188	0.177	3.64E+36	+∞	1.0187	0.034
-0.07	0.15	0.0008	0.014	0.333	0.356	3.24E+71	+∞	1.0188	0.036
-0.02	0.02	0.0002	0.011	0.043	0.022	4.48E-01	3.39E+00	1.0187	0.033
-0.02	0.02	0.0004	0.011	0.043	0.022	4.54E-01	3.42E+00	1.0187	0.033
-0.02	0.02	0.0008	0.011	0.043	0.022	4.64E-01	3.49E+00	1.0187	0.033
-0.02	0.07	0.0002	0.011	0.043	0.022	9.93E-01	3.55E+01	1.0187	0.033
-0.02	0.07	0.0004	0.011	0.044	0.022	1.52E+00	6.65E+01	1.0187	0.033
-0.02	0.07	0.0008	0.011	0.045	0.023	2.49E+00	1.25E+02	1.0187	0.033
-0.02	0.15	0.0002	0.011	0.069	0.048	9.03E+05	+∞	1.0187	0.033
-0.02	0.15	0.0004	0.012	0.096	0.075	4.10E+09	+∞	1.0187	0.034
-0.02	0.15	0.0008	0.013	0.149	0.131	1.40E+17	+∞	1.0188	0.036