

# Auctions with Endogenous Initiation\*

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## Abstract

We study strategic initiation of a first-price auction by potential buyers with changing valuations and the seller. This problem arises in auctions of companies and asset sales, among other contexts. Each buyer can communicate his interest to the seller, thereby triggering an auction. Alternatively, the seller can put the asset for sale without waiting to be approached by a buyer. The bidder's decision to communicate his interest reveals some information about his valuation. In "common-value" auctions, such as battles between financial bidders, it disincentivizes bidders from approaching the seller. In contrast, in "private-value" auctions, such as battles between strategic bidders, the effect is the opposite. Unraveling occurs in the pure "common-value" auctions: no bidder ever approaches the seller, and the auction, if occurs, is seller-initiated. In contrast, equilibria in "private-value" auctions feature both seller- and bidder-initiated auctions. A number of implications about the relation between the initiating party, bids, and auction outcomes are derived and linked to empirical evidence on auctions of companies.

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# 1 Introduction

Over the last several decades, auction theory has developed into a highly influential field with many important results, such as the revenue equivalence theorem.<sup>1</sup> The use of auction theory to model corporate finance transactions, such as mergers and acquisitions and intercorporate asset sales, has also been extensive.<sup>2</sup> To focus on the insights about the auction stage, with very rare exceptions, the literature examines a situation when the asset is already up for sale.

In some cases, exogeneity of an auction taking place is an innocuous assumption. For example, the U.S. Treasury auctions off bonds at a known frequency. In many cases, however, the decision to put the asset for sale is a strategic one. For example, the decision of the board of directors of a firm to sell off a division is an endogenous one. In practice, an auction can be either bidder-initiated, when a potential bidder approaches the seller (e.g., the board of directors of the target company) expressing an interest, in which case the seller then decides to auction the asset off, or seller-initiated, when the seller decides to auction the asset off without being first approached by a potential buyer. To give a flavor of this heterogeneity, consider the following two recent large deals in the M&A market. The acquisition of Taleo, a provider of cloud-based talent management solutions, by Oracle on February 9, 2012 for \$1.9 billion is an example of a bidder-initiated auction. In January 2011, a CEO of a publicly traded technology company, referred in the deal background as Party A, contacted Taleo expressing an interest in acquiring it. Following this contact, Taleo hired a financial adviser that conducted an auction, engaging four more bidders. Oracle was the winning bidder, ending up acquiring Taleo. By contrast, the acquisition of Blue Coat Systems, a provider of Web security, by a private equity firm Thoma Bravo on December 9, 2011 for \$1.1 billion is an example of a seller-initiated takeover auction. In early 2011, Elliot Associates, an activist hedge fund, amassed 9% ownership stake in Blue Coat and forced its board to auction the company. Twelve bidders participated in the auction, and Thoma Bravo was the winner. Overall, not only there exists a considerable heterogeneity with respect to initiator of the contest, but also it appears to be far from

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<sup>1</sup>The formal analysis of auctions goes back to Vickrey (1961). The revenue equivalence theorem was derived by Myerson (1981) and Riley and Samuelson (1981). The overview of results on auction theory can be found, for example, in Krishna (2010).

<sup>2</sup>Among others, see Fishman (1988), Bulow, Huang, and Klemperer (1999), Hansen (2001), and Povel and Singh (2006). Dasgupta and Hansen (2007) provide a review of application of auction theory to corporate finance.

random. For example, in the sample of Fidrmuc et al. (2012), acquisitions by strategic acquirers are more likely to be bidder-initiated, while acquisitions by private equity firms are more likely to be target-initiated.<sup>3</sup>

In this paper, we develop a theory of how potential buyers and the seller choose to initiate auctions. In particular, we ask the following questions: Which characteristics of auctions and the economic environment determine whether auctions are bidder- or seller-initiated? What are the implied inefficiencies? How do bidding strategies and auction outcomes differ depending on how the auction was initiated?

To study these and related questions, we consider a dynamic framework, in which a seller has an asset to sell to one of a number of potential buyers. Each potential buyer has a signal about his valuation of the asset. As time goes by, buyers' valuations may of the asset may change, as they experience shocks, such as changes in the business strategy or in management. Unlike a typical auction model, we deviate from the assumption that the auction takes place at an exogenous date. Here, putting the asset for sale is a strategic decision of the seller. The auction can be initiated by a bidder, when she expresses an interest by sending a message to the seller, which triggers the auction. Alternatively, the auction can be initiated by the seller, when the seller chooses to auction the asset off without first being approached by a bidder. The benefit for the seller to wait is that with some likelihood, a bidder with a high valuation will appear and indicate his interest to the seller, resulting in a better price. Conversely, the benefit to put the asset for sale without waiting to be approached by a bidder is the lack of delay.

The key driving force behind our results is that approaching the seller reveals some information about the bidder's valuation of the asset. Similarly, the lack of any bidder approaching the seller reveals information about valuations of all bidders. If the auction is bidder-initiated, ex-ante identical bidders become endogenously asymmetric at the auction stage: It is common knowledge that the signal of the initiating bidder is drawn from a more optimistic distribution. Other bidders use this information in choosing their bidding strategies. Similarly, the fact that no bidder has approached the seller yet reveals information to a bidder contemplating whether to approach the seller or not that the signals of competitors are low.

We show that the interplay between these information effects heavily depends on the the

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<sup>3</sup>Initiation also appears to be related to characteristics of the seller and auction outcomes (Masulis and Simsir, 2013).

sources of bidders' valuations. In “common-value” auctions, e.g., when a number of private equity shops compete with each other to acquire a poorly-managed target, information effects discourage every bidder from approaching the seller. In pure common-value auctions unraveling occurs in equilibrium: No bidder ever approaches the seller, no matter how high her signal is. All auctions, if they take place, are initiated by the seller. In contrast, the information effect can work in the opposite direction in “private-value” auctions, e.g., when several strategic bidders compete to purchase the asset that they will integrate into their existing operations. In fact, we show that in the stationary equilibrium (i.e., when the distribution of signals, conditional on no auction having taken place in the past, is the same over time) the information effect in “private-value” auctions has the effect on bidder initiation opposite to that in “common-value” auctions: given the same type, a bidder prefers to initiate the auction herself rather than participate in an auction initiated by another bidder.

The intuition behind these results is as follows. Consider a “common-value” setting: all bidders have the same valuation of the asset but differ in their estimates of it. Approaching the seller reveals information to other bidders that the signal of the initiating bidder is sufficiently high: specifically, it is above a certain threshold  $\hat{s}$ . Then, other bidders re-evaluate their estimates of the asset value upwards when they observe that the auction is bidder-initiated. As a result, not only do these bidders update their bids simply due to the fact that they compete against the stronger bidder, but also they update their valuations which results in an even more aggressive bidding. In the case of pure common values, this means that the threshold type  $\hat{s}$  of the initiating bidder only wins the auction, when the types of all rival bidders are the lowest, and in this case she pays the whole value of the target, obtaining zero surplus. Because the argument holds for any  $\hat{s}$ , the only equilibria possible are the ones in which bidder-initiated auctions never happen.

In contrast, the opposite effect takes place in a “private-value” auction. Consider a bidder with a high valuation who contemplates approaching the seller. By observing that no other bidder has approached the seller yet, she learns that valuations of other bidders are likely to be not too high. Approaching the seller immediately allows a bidder with a high valuation to take advantage of a good timing when her rivals are likely to be weak. While other bidders increase their bids in response to competition from the stronger bidder, they do not update their private valuations, which results in money left on the table for

the initiating bidder. In contrast, waiting until another bidder approaches the auction, ensures that a bidder under consideration will compete against a strong rival. Even though participating in an auction initiated by another bidder allows a bidder with a high valuation to hide her type, the former effect always dominates: Competing with a weaker rival, even though she adjusts a bid upwards, is always better than competing with a stronger rival who adjust her bid downwards.

In the “private-value” framework, multiple equilibria can and often do arise. This happens because initiation by bidders and the seller are substitutes. Specifically, if bidders expect the seller to never put herself for sale, they will have strong incentives to approach the seller, because, as argued above, the information effect means that bidders are reluctant to wait until other bidders initiate the auction, as it results in the competition against a strong rival. In contrast, if bidders expect the seller to put herself for sale soon in the future, they will have weak incentives to approach the seller. Intuitively, waiting for a seller to put the asset for sale allows a bidder with a high valuation to hide it from competitors without the cost of competing against a strong rival, which would occur if the bidder waited for another bidder to approach the seller.

Taken together, our theoretical results provide a benchmark with which one can compare empirical findings on initiation of auctions. For example, our results are consistent with empirical evidence on target- and bidder-initiated strategic and private-equity deals, presented in Fidrmuc et. al (2012): approximately 60% (35%) of strategic (private-equity) deals are initiated by the bidders. Our explanation of this large discrepancy is that financial but not strategic bidders have a large common value component in their valuations for targets. Our analysis also has a number of implications about how bids and auction outcomes differ depending on whether it is bidder- or seller-initiated.

Our paper belongs to the vast literature on auction theory. Virtually all the literature already considers a stage when the auction takes place. Two exceptions are recent papers by Cong (2013) and Gorbenko and Malenko (2013), which also feature strategic initiation of an auction but do not study joint initiation by bidders and the seller and focus on bids in cash versus in securities. Both papers assume persistent types and the private-value framework, so the considerations examined here do not occur.

The paper is related to the literature that studies takeover contests as auctions. They have been modeled using both the private-value framework (e.g., Fishman (1988) and

Burkart (1995)) as well as the common-value framework (e.g., Bulow, Huang, and Klemperer (1999), and Povel and Singh (2006)).<sup>4</sup> In our interpretation of the private-value framework as a competition between strategic bidders and the common-value framework as a competition between financial bidders, we follow Bulow, Huang, and Klemperer (1999). None of these papers studies endogenous initiation of takeover contests.

Finally, the paper is related to models of auctions with asymmetric bidders. Most literature on auction theory assumes that bidders are symmetric in the sense that their signals are drawn from the same distribution. Some recent literature (e.g., Maskin and Riley, 2000, 2003; Campbell and Levin, 2000; Lebrun, 2006; Kim, 2008) examines issues that arise when bidders are asymmetric. The novelty of our paper is that asymmetries at the auction stage are not assumed: They arise endogenously and are driven by incentives to approach the seller which differ with the bidder's information. Even though bidders are *ex-ante* identical, meaning that they draw their signals from the same distributions, *ex-interim* at the auction stage they are not. The decision of one bidder to approach the seller and the observation that no other bidder has approached the seller yet endogenously create asymmetry among bidders when the auction takes place.

The remainder of the paper is organized as follows. Section 2 describes the setup of the model. Section 3 studies the “common-values” framework. Section 4 studies the “private-values” framework. Section 5 considers a special case of the private-values model, in which there are two bidders and valuations are distributed uniformly. This special case permits a closed-form solution. Section 6 discusses some implications of the analysis. Section 7 concludes.

## 2 The Model Setup

The economy consists of one risk-neutral seller and  $N > 1$  potential risk-neutral buyers, indexed by  $i = 1, \dots, N$ . The seller has the asset for sale. In the context of application to mergers and intercorporate asset sales, the asset can be the whole company or a business unit. The seller's valuation of the asset is normalized to zero. Time is continuous and indexed by  $t \geq 0$ .

At the initial date  $t = 0$ , each potential buyer  $i$  randomly draws a private signal.

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<sup>4</sup>Bulow and Klemperer (1996, 2009) provide motivations why running a simple auction is often a good way for the seller to sell the asset.

Bidders' signals are independent draws from the uniform distribution over  $[0, \hat{s}_0]$ , where  $\hat{s}_0 \in [0, 1]$ .<sup>5</sup> Conditional on all signals, the value of the asset to bidder  $i$  is  $v(s_i, \mathbf{S}_{-i})$ , where  $\mathbf{S}_{-i} \equiv \{s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_N\}$  is the set of signals of all other bidders.

**Assumption 1.** *Function  $v(s_i, \mathbf{S}_{-i})$  is continuous in all variables, symmetric in the last  $N - 1$  variables, strictly increasing in  $s_i$ , and satisfies  $v(s_i, \mathbf{S}_{-i}) \geq 0 \forall (s_i, \mathbf{S}_{-i}) \in [0, 1]^N$ .*

Assumption 1 is a standard assumption in auction theory. Continuity means that there are no “gaps” in possible valuations of the asset. Symmetry in the last  $N - 1$  variables means that signals of all rival bidders have the same information content for the valuation of a bidder. Strict monotonicity in the first variable means that a higher private signal is always good news about the bidder's valuation. Finally,  $v(s_i, \mathbf{S}_{-i}) \geq 0$  for any combination of signals means that a sale is always the efficient outcome, which is convenient for exposition but not necessary for most results. This valuation structure follows the general symmetric model of Milgrom and Weber (1982). It covers two valuation structures commonly used in the literature:

- **The private-values framework.** This is the case if and only if  $v(s_i, \mathbf{S}_{-i}) = v(s_i)$ . The distinguishing feature of the private values framework is that a bidder's signal provides information only about his own valuation, but not about the valuation of his competitors.
- **The common-values framework.** This is the case if and only if  $v(s_i, \mathbf{S}_{-i}) = v(\mathbf{S})$ , which is symmetric in all  $N$  variables. Conditional on all signals, all bidders have the same valuation of the asset. However, bidders differ in their assessment of the value of the asset, because their private signals are different.

We focus on these two valuation structures. There are two natural interpretations of common values versus private values in the context of auctions of companies and business units. The first interpretation deals with different types of bidders. Following Bulow, Huang, and Klemperer (1999), we can interpret the common-value (private-value) auction as a battle between two financial (strategic) bidders. Intuitively, financial bidders tend to use

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<sup>5</sup>Because we assume a general functional form that maps signals into valuations, uniform distribution is, to a large extent, a normalization.

the same strategies after they acquire the target (i.e., have “common” values), but may have different estimates of potential gains (i.e., have different signals about the common value). In contrast, because synergies that one strategic bidder expects to achieve from acquiring the target are often bidder-specific, they provide little information about valuation of the target to the other bidder. This interpretation is somewhat consistent with the finding of Gorbenko and Malenko (2012) that conditional on observed characteristics of the target, valuations of strategic bidders are more dispersed than valuations of financial bidders. The second interpretation deals with different types of targets rather than bidders. Broadly, value can be created in acquisitions either because the incumbent management of the target is inefficient or because the target and the acquirer have synergies that cannot be realized by the acquirer as a stand-alone company. Acquisitions of the first type are common-value deals, while acquisitions of the second type are private-value deals.

In practice, the environment changes over time, as either the business nature or management of a bidder or a target changes, or external economic shocks arrive. We capture this feature in the following way. As time goes by, each bidder can experience a shock. A shock arrives according to the Poisson process with intensity  $\lambda > 0$ , and shocks are independent across bidders. If bidder  $i$  experiences a shock, he draws a new signal  $s_i$  from uniform distribution over  $[0, 1]$ . His valuation of the asset changes to  $v(s_i, \mathbf{S}_{-i})$ , and his previous signal becomes irrelevant.<sup>6</sup> The general idea behind this assumption is that there is an option value in not acquiring the asset today if the valuation is positive but low.

The seller has the right to auction the asset among the bidders at any time. By putting the asset for sale at time  $t$ , the seller commits to sell the asset through a sealed-bid first-price auction with no reserve price. Specifically, each bidder simultaneously submits a bid to the seller in a concealed fashion. The  $N$  bids are compared, and the bidder that submitted the highest bid acquires the asset and pays the bid she submitted. Once the auction takes place, the game is over. The winning bidder obtains the payoff that equals to her valuation less the price she pays. All losing bidders obtain zero payoffs. The seller obtains the payoff that equals to the winning bid.

Prior to the auction, each bidder may communicate privately with the seller by sending a message signaling his interest in acquiring the asset. Communication is costless and follows Crawford and Sobel (1982) with the binary message space (0 or 1). Without loss

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<sup>6</sup>Alternatively, we could assume that an existing bidder exits the game and obtains some exit payoff  $X$ , while a new bidder comes in.



of generality, message  $m = 1$  is interpreted as “I am interested,” and message  $m = 0$  is interpreted as the lack of communication. As we will see, in equilibrium, upon receiving message  $m = 1$ , the seller will auction the asset off immediately. We refer to such an event as a “*bidder-initiated*” auction, capturing the fact that the auction is triggered by a bidder communicating his interest to the seller. The seller may also auction the asset off without receiving message  $m = 1$ . We refer to such an event as a “*seller-initiated*” auction, capturing the fact that the auction was not triggered by any bidder communicating his interest to the seller. After the decision to undertake an auction but prior to bidders’ submitting bids, the seller may disclose whether the auction is bidder- or seller-initiated. The seller does not have to disclose it but cannot lie.<sup>7</sup> By the standard reasoning (Grossman, 1981; Milgrom, 1981), since it is common knowledge that the seller knows if the auction is bidder- or seller-initiated, the seller will always disclose it.

## 2.1 The equilibrium concept

The equilibrium concept is Markov Perfect Bayesian Equilibrium (MPBE). At the auction, the strategy of each bidder is a mapping from his own signal and the belief about the signals of other bidders into a non-negative bid. Prior to the auction, the strategy of each bidder is a mapping from his own signal and the belief about the signals of other bidders into  $m \in \{0, 1\}$ . The strategy of the seller is a mapping from his belief about the signals of all bidders into the decision whether to auction the asset or wait. Because bidders are ex-ante symmetric, we look for equilibria in which the bidders follow symmetric strategies prior to the auction. Furthermore, we look for equilibria in which at any time  $t$  prior to the auction a bidder follows the cut-off communication strategy, such that a bidder sends message  $m = 1$  if and only if his signal is above some cut-off  $\hat{s}_t$ .<sup>8</sup>

For much of the paper, we consider the stationary case, defined as the situation in

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<sup>7</sup>Lying can be punished either implicitly because of reputational concerns of the investment bank advising the seller or explicitly: If the seller is a publicly-traded US corporation, the seller is obligated to disclose the sale process as part of the “background of the merger” (e.g., Boone and Mulherin, 2007); if some information there is not truthful, the seller might get sued.

<sup>8</sup>Because arbitrarily low types always obtain an arbitrarily low surplus from the auction, it is straightforward that there is no equilibrium in which low types send the message that triggers the auction, while high types do not. What is less clear, however, is whether there are equilibria in which communication strategies are not described by a cut-off (e.g., if there can be multiple cut-offs). Because the analysis of first-price auctions when distributions of valuations have an arbitrary number of gaps is, to our knowledge, an open problem, we cannot characterize the whole set of symmetric MPBE.

which the cut-off  $\hat{s}_t$  stays constant over time at some level  $\hat{s}$ . This requires that the initial condition is  $\hat{s}_0 = \hat{s}$ . Because in practice there is often no clear starting date, at least, in the applications we look at, focusing on the stationary solution is reasonable. Towards the end of the paper, we provide an analysis of the non-stationary dynamics, starting at  $\hat{s}_0 = 1$ .

### 3 The Case of Common Values

Consider the case of pure common values,  $v(s_i, \mathbf{S}_{-i}) = v(\mathbf{S})$ , where  $v(\mathbf{S})$  is symmetric in all  $N$  variables.

#### 3.1 Equilibria in Bidder- and Seller-Initiated Auctions

First, we solve for the equilibrium at the auction stage.

##### 3.1.1 A bidder-initiated auction

Consider a bidder-initiated auction with an exogenous cut-off type  $\hat{s}$ . Without loss of generality, denote the initiating bidder by bidder 1. Then, from the point of view of other bidders and the seller, the type of the initiating bidder is distributed uniformly over  $[\hat{s}, 1]$ . By contrast, the type of each non-initiating bidder is distributed uniformly over  $[0, \hat{s}]$ . Thus, even though all bidders are ex-ante symmetric, initiation endogenously creates an asymmetry between the initiating bidder and all other bidders.

Conjecture that the equilibrium in pure strategies exists. Let  $a_I(s, \hat{s})$  denote the equilibrium bid of the initiating bidder with signal  $s_1 \geq \hat{s}$ , given that all other bidders believe that types  $\hat{s}$  and above initiate the contest. Similarly, let  $a_N(s, \hat{s})$  denote the equilibrium bid of the non-initiating bidder with signal  $s$ . Let  $\bar{a}(\hat{s}) \equiv a_I(1, \hat{s}) = a_N(\hat{s}, \hat{s})$  be the common highest bid submitted by both bidders.<sup>9</sup> Consider the initiating bidder with signal  $s$  and bid  $b$ . The expected payoffs of the initiating bidder and each non-initiating bidder with signal  $s$  and bid is  $b$  are

$$P_I(b, s, \hat{s}) = \int_0^{\phi_N(b, \hat{s})} \dots \int_0^{\phi_N(b, \hat{s})} (v(s, x_2, \dots, x_{N-1})) dx_2 \dots dx_N.$$

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<sup>9</sup>The proof that  $a_I(1, \hat{s}) = a_N(\hat{s}, \hat{s})$  is straightforward. Suppose  $a_I(1, \hat{s}) > a_N(\hat{s}, \hat{s})$ . Then, types of bidder 1 close enough to 1 can reduce their bids and still win the auction with probability 1. Thus,  $a_I(1, \hat{s}) > a_N(\hat{s}, \hat{s})$  cannot occur in equilibrium. Similarly,  $a_I(1, \hat{s}) < a_N(\hat{s}, \hat{s})$  cannot occur in equilibrium.

$$P_I(b, s, \hat{s}) = \int_0^{\phi_N(b, \hat{s})} \dots \int_0^{\phi_N(b, \hat{s})} (v(s, x_2, \dots, x_N) - b) \frac{dx_2 \dots dx_N}{\hat{s}^{N-1}}, \quad (1)$$

$$P_N(b, s, \hat{s}) = \int_{\hat{s}}^{\phi_I(b, \hat{s})} \int_0^{\phi_N(b, \hat{s})} \dots \int_0^{\phi_N(b, \hat{s})} (v(s, x_2, x_3, \dots, x_N) - b) \frac{dx_2 \dots dx_N}{(1 - \hat{s}) \hat{s}^{N-2}}. \quad (2)$$

where  $\phi_j \equiv a_j^{-1}$ ,  $j \in \{I, N\}$  is the inverse of the bidding function. Intuitively, the expected payoff of a bidder from the auction equals her valuation, which depends on the realization of the competitors' signals, less her bid, integrated over all realizations of the competitor's signal, such that the bidder is the winner.

Taking the first-order conditions of (1) and (2), we get

$$0 = (N-1) \frac{\partial \phi_N(b, \hat{s})}{\partial b} \int_0^{\phi_N(b, \hat{s})} \dots \int_0^{\phi_N(b, \hat{s})} (v(s, \phi_N(b, \hat{s}), x_3, \dots, x_N) - b) dx_3 \dots dx_N - \phi_N(b, \hat{s})^{N-1}, \quad (3)$$

$$\begin{aligned} 0 = & \frac{\partial \phi_I(b, \hat{s})}{\partial b} \int_0^{\phi_N(b, \hat{s})} \dots \int_0^{\phi_N(b, \hat{s})} (v(s, \phi_I(b, \hat{s}), x_3, \dots, x_N) - b) dx_3 \dots dx_N \\ & + (N-2) \frac{\partial \phi_N(b, \hat{s})}{\partial b} \int_{\hat{s}}^{\phi_I(b, \hat{s})} \int_0^{\phi_N(b, \hat{s})} \dots \int_0^{\phi_N(b, \hat{s})} (v(s, \phi_N(b, \hat{s}), x_3, \dots, x_N) - b) dx_3 \dots dx_N \\ & - (\phi_I(b, \hat{s}) - \hat{s}) \phi_N(b, \hat{s})^{N-2}. \end{aligned} \quad (4)$$

In equilibrium,  $b = a_I(s, \hat{s})$  must satisfy (3), implying  $s = \phi_I(b, \hat{s})$ , and  $b = \beta_N(s, \hat{s})$  must satisfy (4), implying  $s = \phi_N(b, \hat{s})$ . Plugging in and rearranging the terms, we get that for all  $b < \bar{b}$  the following differential equations must hold:

$$\begin{aligned} (N-1) \frac{\partial \phi_I(b, \hat{s})}{\partial b} &= \frac{\phi_N(b, \hat{s})}{\mathbb{E}[v(\phi_I(b, \hat{s}), \phi_N(b, \hat{s}), x_3, \dots, x_N) | x_3, \dots, x_N \leq \phi_N(b, \hat{s})] - b}; \\ 1 &= \frac{\frac{\partial \phi_I(b, \hat{s})}{\partial b} \mathbb{E}[v(\phi_N(b, \hat{s}), \phi_I(b, \hat{s}), x_3, \dots, x_N) | x_3, \dots, x_N \leq \phi_N(b, \hat{s})] - b}{\phi_I(b, \hat{s}) - \hat{s}} \\ &+ (N-2) \frac{\frac{\partial \phi_N(b, \hat{s})}{\partial b} \mathbb{E}[v(\phi_N(b, \hat{s}), \phi_N(b, \hat{s}), x_3, \dots, x_N) | x_3 \in [\hat{s}, \phi_I(b, \hat{s})], x_4, \dots, x_N \leq \phi_N(b, \hat{s})] - b}{\phi_N(b, \hat{s})} \end{aligned} \quad (5)$$

The intuition behind (5)–(6) is as follows. Bidder  $i$  faces a trade-off. On one hand, by increasing her bid from  $b$  a little bit, she increases the likelihood of winning. In this marginal event, she will outbid one of the other bidders only by a little bit. If bidder  $i$  is the ini-

tiating bidder, it means that in the marginal event one of the other  $N - 1$  bidders has a signal of exactly  $\phi_N(b, \hat{s})$ . Thus, in the marginal event, bidder  $i$  gets the surplus of  $\mathbb{E}[v(\phi_I(b, \hat{s}), \phi_N(b, \hat{s}), x_3, \dots, x_N) | x_3, \dots, x_N \leq \phi_N(b, \hat{s})] - b$ . If bidder  $i$  is a non-initiating bidder, then there are two types of marginal events: either she outbids the initiating bidder with signal  $\phi_I(b, \hat{s})$  or she outbids one of the other non-initiating bidders with signal  $\phi_N(b, \hat{s})$ . On the other hand, by increasing her bid from  $b$  a little bit, bidder  $i$  will pay more whenever he wins the auction. In equilibrium, the mapping between bids and signals is such that the marginal costs and benefits are equal, yielding (5)–(6).

The system of equations (5)–(6) is solved subject to the boundary conditions, which are to be determined. Let  $[\underline{a}(\hat{s}), \bar{a}(\hat{s})]$  denote the interval of possible bids. It must be the same for both bidders, as otherwise one of the bidders finds it optimal to deviate. The upper boundary implies  $1 = \phi_I(\bar{a}(\hat{s}), \hat{s})$  and  $\hat{s} = \phi_N(\bar{a}(\hat{s}), \hat{s})$ . Consider the lower boundary  $\underline{a}(\hat{s})$ . First, it cannot be below  $v(\hat{s}, \mathbf{0})$ , because either the initiating bidder of type  $\hat{s}$  or a non-initiating bidder of type 0 would find it optimal to deviate and submit a marginally higher bid. By doing this, she can increase the probability of winning from zero to a positive number, and thus get a positive expected surplus instead of zero. Therefore,  $\underline{b}(\hat{s}) \geq v(\hat{s}, \mathbf{0})$ . Second,  $\underline{b}(\hat{s})$  cannot be above  $v(\hat{s}, \mathbf{0})$ , because such  $\underline{b}(\hat{s})$  would imply that low enough types get a negative surplus in equilibrium. Thus, we get the following lemma:

**Lemma 1 (equilibrium in the bidder-initiated CV auction).** *The equilibrium bidding strategies of the initiating and non-initiating bidders,  $\alpha_I(s, \hat{s})$  and  $a_N(s, \hat{s})$ , are increasing functions, such that their inverses satisfy (5)–(6), with boundary conditions*

$$1 = \phi_I(\bar{a}(\hat{s}), \hat{s}), \quad (7)$$

$$\hat{s} = \phi_N(\bar{a}(\hat{s}), \hat{s}) \quad (8)$$

$$\underline{a}(\hat{s}) = v(\hat{s}, \mathbf{0}). \quad (9)$$

In particular, for the special case of  $N = 2$ , differential equations 5)–(6) take a simple

form:

$$\frac{\partial \phi_N(b, \hat{s})}{\partial b} = \frac{\phi_N(b, \hat{s})}{v(\phi_I(b, \hat{s}), \phi_N(b, \hat{s})) - b}, \quad (10)$$

$$\frac{\partial \phi_I(b, \hat{s})}{\partial b} = \frac{\phi_I(b, \hat{s}) - \hat{s}}{v(\phi_I(b, \hat{s}), \phi_N(b, \hat{s})) - b}. \quad (11)$$

Furthermore, if the valuation is additive,  $v(s_1, s_2) = \frac{1}{2}(s_1 + s_2)$ , then the equilibrium bidding strategies are linear:

$$a_I(s, \hat{s}) = \frac{s + \hat{s}(1 - 2\hat{s})}{4(1 - \hat{s})},$$

$$a_N(s, \hat{s}) = \frac{s + 2\hat{s}^2}{4\hat{s}}.$$

The common range of bids is  $[\frac{\hat{s}}{2}, \frac{1+2\hat{s}}{4}]$ . The equilibrium bids are plotted on the left-panel of Figure 1 for the case  $\hat{s} = 0.5$ .

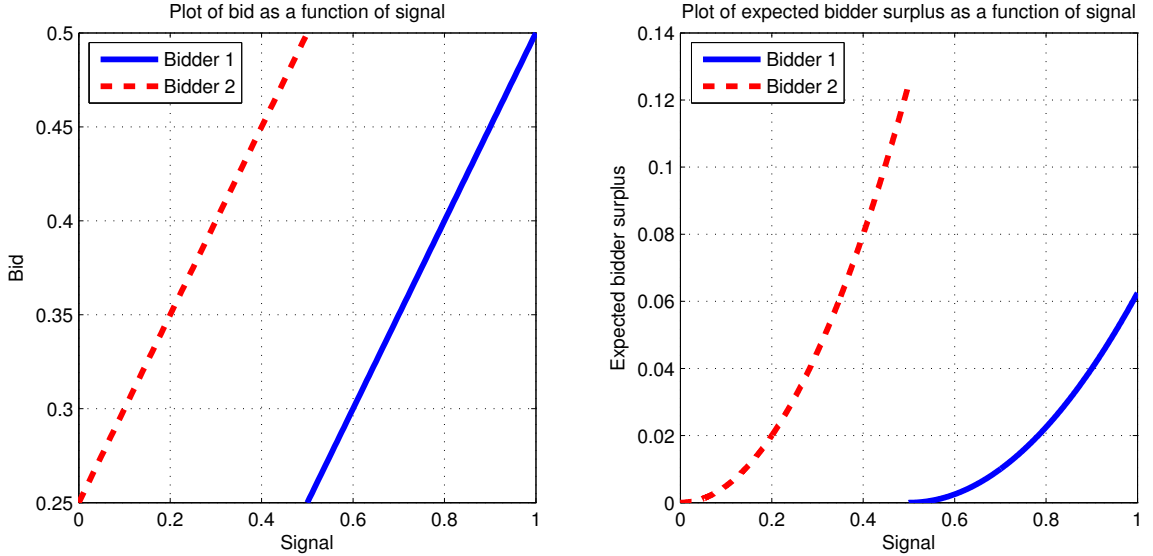


Figure 1: **Equilibrium bids and expected payoffs of bidders in a bidder-initiated common-value auction.** The left panel plots the equilibrium bids as functions of signals for the initiating bidder (the blue normal line) and the other bidder (the red dashed line). The right panel plots the corresponding expected surpluses of each bidder.

The equilibrium in the auction implies the payoff of type  $\hat{s}$  of the initiating bidder is zero:

**Lemma 2.** *The equilibrium payoff of the initiating bidder of type  $\hat{s}$  is zero:  $P_I(\underline{a}(\hat{s}), \hat{s}, \hat{s}) = 0$ .*

**Proof.** Follows immediately from  $\underline{a}(\hat{s}) = v(\hat{s}, \mathbf{0})$ .

The intuition behind the lemma is as follows. All non-initiating bidders know that the initiating bidder approaches the seller if and only if her signal is at least  $\hat{s}$ . Therefore, when a non-initiating bidder sees that the auction was bidder-initiated, she re-evaluates the target to at least  $v(s, \hat{s}, \mathbf{0})$ , where  $s$  is her own signal. Similarly, the initiating bidder estimates that the target is worth at least  $v(s, \mathbf{0})$ , where  $s$  is her signal. Because under no circumstance the target is valued less than  $v(\hat{s}, \mathbf{0})$ , no bidder in equilibrium bids less than this. However, type  $\hat{s}$  of bidder 1 wins the auction only when the signals of all other bidders are zeros, i.e., when all other bidders submit  $\underline{a}(\hat{s})$ . In this situation, the value of the asset is exactly  $v(\hat{s}, \mathbf{0})$ , leaving bidder 2 without surplus. Note that this result holds for any cut-off type  $\hat{s}$ .

An important result from the bidding stage is that the initiating bidder obtains a zero expected payoff. As we show in the next section, this result is unique to the common-value setting. The argument behind this result generalizes the logic of Engelbrecht-Wiggans, Milgrom, and Weber (1983) who show that a bidder that has access to only public information always gets zero surplus in equilibrium. Here, the initiating bidder does have proprietary information, because her decision to approach the target only reveals that her signal is above a certain level. Therefore, she does get a positive expected surplus, unlike a bidder with public information in Engelbrecht-Wiggans, Milgrom, and Weber (1983). This can be seen on the right panel of Figure 1. Despite this, the *marginal* type of the initiating bidder,  $\hat{s}$ , gets zero surplus, because her decision to approach the seller makes it common knowledge that her type is at least  $\hat{s}$ . Put differently, in the common-values model, the bidder gets expected surplus because of her information rents: To reveal her higher signal through a higher bid, the bidder must be compensated with a higher surplus, which takes the form of a higher probability of winning. However, type  $\hat{s}$  of bidder 1 has no information rent: It is common knowledge that  $s_1$  is at least  $\hat{s}$ , so type  $\hat{s}$  has no lower types to separate from; hence, there is no need to pay her.

### 3.1.2 A seller-initiated auction

Suppose that the seller initiates the auction. Conditional on no bidder approaching the seller, all parties believe that each bidder's signal is a random draw from  $[0, \hat{s}]$  for some cut-off type  $\hat{s}$ . Because the auction is seller-initiated, all bidders are symmetric, so we will look for an equilibrium in symmetric bidding strategies. Let  $\alpha(s, \hat{s})$  denote the bid of a bidder with signal  $s$ . The expected payoff of a bidder with signal  $s$  who bids  $b$  is

$$P(b, s, \hat{s}) = \int_0^{\phi(b, \hat{s})} \dots \int_0^{\phi(b, \hat{s})} (v(s, x_2, \dots, x_N) - b) \frac{dx_2 \dots dx_N}{\hat{s}^{N-1}}, \quad (12)$$

where  $\phi(b, \hat{s})$  is the inverse in  $s$  of  $\alpha(s, \hat{s})$ . Taking the first-order condition and using the fact that the maximum is reached at  $b = \alpha(s, \hat{s})$  (or, equivalently,  $s = \phi(b, \hat{s})$ ), we obtain differential equation

$$(N-1) \frac{\partial \phi(b, \hat{s})}{\partial b} = \frac{\phi(b, \hat{s})}{\mathbb{E}[v(\phi(b, \hat{s}), \phi(b, \hat{s}), x_3, \dots, x_N) | x_3, \dots, x_N \leq \phi(b, \hat{s})] - b}. \quad (13)$$

This equation is solved subject to the boundary condition  $\beta(0, \hat{s}) = v(\mathbf{0})$ , or, equivalently,  $0 = \phi(v(\mathbf{0}), \hat{s})$ . Intuitively, the lowest type wins the auction only if all her competitors also have the lowest signals. The value of the asset conditional on this event is  $v(\mathbf{0})$ . Because under no circumstance the target is valued less than  $v(\mathbf{0})$ , the bidders compete away the value and bid  $v(\mathbf{0})$ . Note that neither (13) nor the initial value condition depend on  $\hat{s}$  other than through  $\phi(\cdot)$ . Therefore, the solution is independent of  $\hat{s}$ . We denote it by  $\phi(b)$  and  $\beta(s)$ . To summarize:

**Lemma 3 (equilibrium in the seller-initiated CV auction).** *If the seller initiates the auction, all  $N$  bidders are symmetric. The symmetric equilibrium bidding strategies are independent of  $\hat{s}$  and solve (13) subject to the initial value condition  $0 = \phi(v(\mathbf{0}), \hat{s})$  ( $\beta(0, \hat{s}) = v(\mathbf{0})$ ).*

When  $N = 2$ , the equilibrium simplifies to the well-known result:

$$\beta(s) = \mathbb{E}[v(x, x) | x \leq s]. \quad (14)$$

Importantly, all bidders except for the lowest type 0 obtain positive expected payoff

from the auction. In particular, the cut-off type  $\hat{s}$  obtains a positive payoff.

### 3.2 The Initiation Game

Because the marginal type of the initiating bidder always obtains zero surplus in equilibrium, it is straightforward to show the unraveling result: bidder initiation never happens. To see this, suppose by contradiction that  $\hat{s}_t < 1$  at some time  $t$ . Then, types  $s \rightarrow \hat{s}_t$  obtain an infinitesimal payoff by approaching the seller. Consider a deviation in which type  $s$ , close enough to  $\hat{s}_t$ , never approaches the seller. The payoff from this deviation is strictly positive and bounded away from zero, because all  $P(s, \hat{s})$ ,  $P_I(s, \hat{s})$ , and  $P_N(s, \hat{s})$  are positive and bounded away from zero. Thus, types  $s \rightarrow \hat{s}_t$  strictly benefit from waiting, so there exists no equilibrium in which  $\hat{s}_t < 1$  at some time  $t$ .

By contrast, there always exists an equilibrium in which the seller immediately initiates the auction. It is also the unique equilibrium, because waiting is never optimal for the seller when no bidder ever approaches the seller:

**Proposition 1.** *There exists a unique equilibrium. In this equilibrium, no bidder ever approaches the seller, and the seller initiates the auction immediately at  $t = 0$ .*

It is straightforward to extend the model by assuming that running an auction costs  $I > 0$  to the seller. The following corollary illustrates the unique equilibrium in this case:

**Corollary.** *Let*

$$\hat{I}^{cv} \equiv \mathbb{E} \left[ \beta \left( \max_{i \in \{1, \dots, N\}} s_i \right) \mid s_i \in [0, 1] \ \forall i \right]. \quad (15)$$

*If  $I < \hat{I}^{cv}$ , the seller initiates the auction at the initial date. If  $I > \hat{I}^{cv}$ , the seller does not initiate the auction, no bidder approaches the seller, and the sale never happens.*

## 4 The Case of Private Values

Next, we consider the case of private values: bidder  $i$  has the valuation of  $v(s_i)$ . First, we solve for the equilibria in bidder- and seller-initiated auctions. Then, we consider the initiation game, taking the equilibria in the auctions as given.



In the analysis of the auction, we impose the following restriction on equilibrium bids:

**Assumption 2.** *No bidder bids more than his valuation in equilibrium.*

The rationale behind this assumption is that bidding above one's valuation is a dominated strategy. Still, as Kaplan and Zamir (2011) show, if no such restriction is imposed, multiple equilibria in the first-price auction with asymmetric bidders arise, in which some bidders submit “non-serious” bids (i.e., bids that win with probability zero) above their valuations. Restricting equilibrium bids to be below valuations pins down the unique equilibrium in the auction (Lebrun, 2006).

## 4.1 Equilibria in Bidder- and Seller-Initiated Auctions

### 4.1.1 A bidder-initiated auction

Suppose that a bidder approaches the seller if and only if his new signal exceeds  $\hat{s}$ . Without loss of generality, denote the initiating bidder by bidder 1. The other bidders are denoted by index  $i = 2, \dots, N$ . Thus, at the auction stage bidder 1 believes that the types of other bidders are distributed uniformly over  $[0, \hat{s}]$ . Each of the other  $N - 1$  bidders believes that the types of  $N - 2$  of her competitors are distributed uniformly over  $[0, \hat{s}]$ , and the type of the remaining bidder is distributed uniformly over  $[\hat{s}, 1]$ . Denote the equilibrium bid of the initiating bidder by  $\beta_I(s_i, \hat{s})$ , and the equilibrium bids of the other bidders by  $\beta_N(s_i, \hat{s})$ . We denote the corresponding inverses by  $\gamma_I(b, \hat{s})$  and  $\gamma_N(b, \hat{s})$ .

By the standard argument (e.g., Chapter 4.2 in Krishna, 2010), the minimum and maximum “serious” bids (i.e., bids that win with a positive probability) of both the initiating and the non-initiating bidders must be the same. Denote them by  $\underline{b}(\hat{s})$  and  $\bar{b}(\hat{s})$ . The expected payoff of the initiating bidder with signal  $s \in [\hat{s}, 1]$  and bid  $b$  is

$$\begin{aligned} \Pi_I(b, s, \hat{s}) &= \Pr(\beta_N(s_2, \hat{s}) \leq b)^{N-1} (v(s) - b) \\ &= \left( \frac{\gamma_N(b, \hat{s})}{\hat{s}} \right)^{N-1} (v(s) - b). \end{aligned} \tag{16}$$

Intuitively, if this bidder makes a bid  $b$ , it exceeds the bid of each of its competitors with probability  $\gamma_N(b, \hat{s})/\hat{s}$ . Thus, the bid of  $b$  leads to winning the auction with probability

$(\gamma_N(b, \hat{s})/\hat{s})^{N-1}$ , conditional on which the bidder gets the payoff of his valuation of the asset less his payment. The first-order condition is

$$(N-1) \frac{\partial \gamma_N(b, \hat{s})}{\partial b} (v(s) - b) = \gamma_N(b, \hat{s}). \quad (17)$$

In equilibrium, the optimal bid is given by  $\beta_I(s, \hat{s})$ , implying that  $s = \gamma_I(b, \hat{s})$ . Thus,

$$(N-1) \frac{\partial \gamma_N(b, \hat{s})}{\partial b} (v(\gamma_I(b, \hat{s})) - b) = \gamma_N(b, \hat{s}). \quad (18)$$

This expression equates the benefits of the initiating bidder from increasing his bid by a small margin (the higher probability of winning - the left-hand side) with the cost of paying more conditional on winning (the right-hand side).

Similarly, the expected payoff of the non-initiating bidder with signal  $s \in [0, \hat{s}]$  and bid  $b$  is

$$\begin{aligned} \Pi_N(b, s, \hat{s}) &= \Pr(\beta_I(s_1, \hat{s}) \leq b) \Pr(\beta_N(s_2, \hat{s}) \leq b)^{N-2} (v(s) - b) \\ &= \frac{\gamma_I(b, \hat{s}) - \hat{s}}{1 - \hat{s}} \left( \frac{\gamma_N(b, \hat{s})}{\hat{s}} \right)^{N-2} (v(s) - b). \end{aligned} \quad (19)$$

Intuitively, if a non-initiating bidder bids  $b$ , his bid exceeds the bid of the initiating bidder with probability  $(\gamma_I(b, \hat{s}) - \hat{s}) / (1 - \hat{s})$ , and the bid of each single non-initiating bidder with probability  $\gamma_N(b, \hat{s}) / \hat{s}$ . Thus, the probability of winning is  $\frac{\gamma_I(b, \hat{s}) - \hat{s}}{1 - \hat{s}} \left( \frac{\gamma_N(b, \hat{s})}{\hat{s}} \right)^{N-2}$ , conditional on which the bidder gets the payoff of his valuation  $v(s)$  less his payment  $b$ . The first-order condition is

$$\begin{aligned} &\frac{\partial \gamma_I(b, \hat{s}) / \partial b}{1 - \hat{s}} \left( \frac{\gamma_N(b, \hat{s})}{\hat{s}} \right)^{N-2} (v(s) - b) + \frac{\gamma_I(b, \hat{s}) - \hat{s}}{1 - \hat{s}} (N-2) \left( \frac{\gamma_N(b, \hat{s})}{\hat{s}} \right)^{N-3} \frac{\partial \gamma_N(b, \hat{s}) / \partial b}{\hat{s}} (v(s) - b) \\ &= \frac{\gamma_I(b, \hat{s}) - \hat{s}}{1 - \hat{s}} \left( \frac{\gamma_N(b, \hat{s})}{\hat{s}} \right)^{N-2}. \end{aligned} \quad (20)$$

Equivalently,

$$\left( \frac{\partial \gamma_I(b, \hat{s})}{\partial b} \gamma_N(b, \hat{s}) + (N-2) \frac{\partial \gamma_N(b, \hat{s})}{\partial b} (\gamma_I(b, \hat{s}) - \hat{s}) \right) (v(s) - b) = (\gamma_I(b, \hat{s}) - \hat{s}) \gamma_N(b, \hat{s}). \quad (21)$$

In equilibrium, the optimal bid is  $b = \beta_N(s, \hat{s})$ . Equivalently,  $s = \gamma_N(b, \hat{s})$ . Thus,

$$\left( \frac{\partial \gamma_I(b, \hat{s})}{\partial b} \gamma_N(b, \hat{s}) + (N-2) \frac{\partial \gamma_N(b, \hat{s})}{\partial b} (\gamma_I(b, \hat{s}) - \hat{s}) \right) (v(\gamma_N(b, \hat{s})) - b) = (\gamma_I(b, \hat{s}) - \hat{s}) \gamma_N(b, \hat{s}). \quad (22)$$

This expression equates the marginal benefits of increasing the bid by a non-initiating bidder in the higher probability of winning (the left-hand side) with the marginal cost of the higher payment conditional on winning (the right-hand side).

The system of two equations, (18) and (22), is solved subject to the following boundary conditions:

$$\gamma_I(\underline{b}(\hat{s}), \hat{s}) = \hat{s}, \quad (23)$$

$$\gamma_N(\underline{b}(\hat{s}), \hat{s}) = v^{-1}(\underline{b}(\hat{s})), \quad (24)$$

$$\gamma_I(\bar{b}(\hat{s}), \hat{s}) = 1, \quad (25)$$

$$\gamma_N(\bar{b}(\hat{s}), \hat{s}) = \hat{s}. \quad (26)$$

Condition (23) means that the lowest type of the initiating bidder,  $\hat{s}$ , submits the lowest serious bid. Condition (24) means that type of a non-initiating bidder that submits the lowest serious bid bids her valuation. Intuitively, otherwise either this bidder would bid above his valuation, which violates Assumption 2, or would profitably deviate to increasing her bid a little, which would result in a positive expected payoff, exceeding her equilibrium payoff of zero. Conditions (25)–(26) simply mean that the highest bid is submitted by the highest possible types of the initiating and the non-initiating bidders.

Assumption 2 pins down the minimum bid (see Lebrun (2006) for a proof). Specifically, this condition gives the lowest bid  $\underline{b}(\hat{s})$  to maximize

$$\underline{b}(\hat{s}) \in \arg \max_b \left( \frac{v^{-1}(b)}{\hat{s}} \right)^{N-1} (v(\hat{s}) - b), \quad (27)$$

which yields

$$(N-1) \frac{v(\hat{s}) - \underline{b}(\hat{s})}{v'(v^{-1}(\underline{b}(\hat{s})))} = v^{-1}(\underline{b}(\hat{s})). \quad (28)$$

The following lemma summarizes the unique equilibrium in the bidder-initiated first-price auction. Existence and uniqueness results follow from Lebrun (2006).

**Lemma 4 (equilibrium in the bidder-initiated PV auction).** *There exists a unique (up to the non-serious bids of types  $s < v^{-1}(\underline{b}(\hat{s}))$  of non-initiating bidders) equilibrium in the bidder-initiated auction. The inverse bidding functions of the initiating and non-initiating bidders,  $\gamma_I(b, \hat{s})$  and  $\gamma_N(b, \hat{s})$ , satisfy equations (18) and (22) with boundary conditions (23)–(26) and the lowest serious bid is given by (27).*

Figure 2 illustrates the equilibrium bidding strategies for the case  $N = 2$ ,  $v(s) = s$ , and  $\hat{s} = 0.5$ :

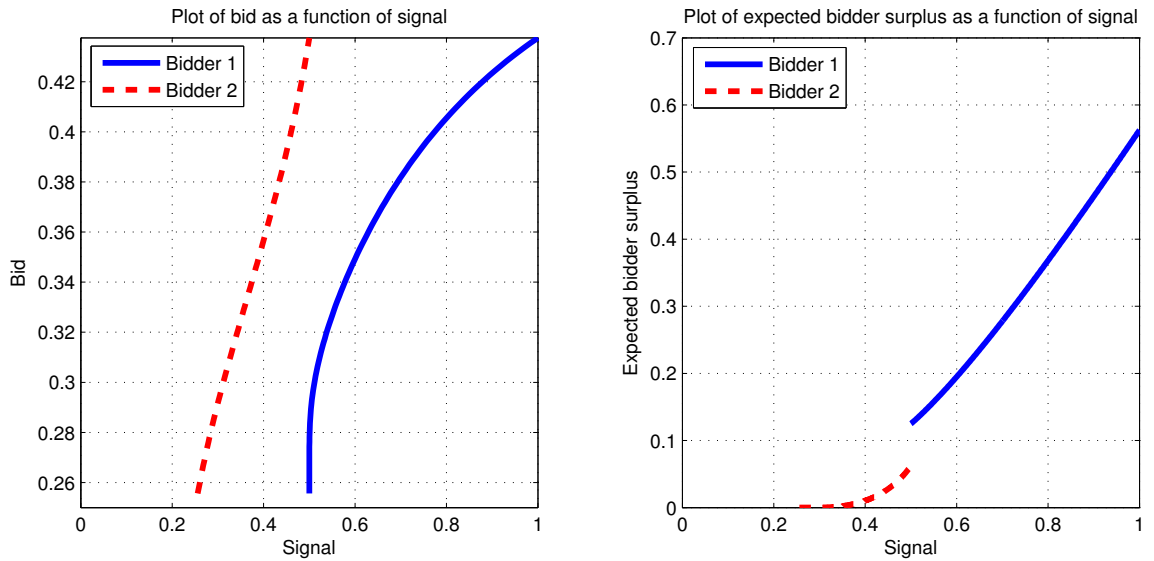


Figure 2: **Equilibrium bids and expected payoffs of bidders in a bidder-initiated private-value auction.** The left panel plots the equilibrium bids as functions of signals for the initiating bidder (the blue normal line) and the other bidder (the red dashed line). The right panel plots the corresponding expected surpluses of each bidder.

We denote  $\Pi_I(s, \hat{s}) = \Pi_I(\beta_I(s, \hat{s}), s, \hat{s})$  and  $\Pi_N(s, \hat{s}) = \Pi_N(\beta_N(s, \hat{s}), s, \hat{s})$ . The next lemma shows that the payoff of the marginal type,  $\hat{s}$ , is higher if he is the one initiating the auction than if the auction is initiated by another bidder:

**Lemma 5.** *For any  $\hat{s}$ ,  $\Pi_I(\hat{s}, \hat{s}) \geq \Pi_N(\hat{s}, \hat{s})$  in equilibrium.*

**Proof.** The equilibrium payoff of the initiating bidder with type  $\hat{s}$  is

$$\begin{aligned}
\Pi_I(\hat{s}, \hat{s}) &= \left( \frac{\gamma_N(\underline{b}(\hat{s}), \hat{s})}{\hat{s}} \right)^{N-1} (v(\hat{s}) - \underline{b}(\hat{s})) \\
&= \max_{b \in [\underline{b}, \bar{b}]} \left( \frac{\gamma_N(b, \hat{s})}{\hat{s}} \right)^{N-1} (v(\hat{s}) - b) \\
&\geq \left( \frac{\gamma_N(\bar{b}, \hat{s})}{\hat{s}} \right)^{N-1} (v(\hat{s}) - \bar{b}) \\
&= v(\hat{s}) - \bar{b} = \Pi_N(\hat{s}, \hat{s}).
\end{aligned}$$

Therefore,  $\Pi_I(\hat{s}, \hat{s}) \geq \Pi_N(\hat{s}, \hat{s})$ .

This result is in stark contrast with the case of common values, in which the boundary type of the initiating bidder always obtains zero expected payoff, which, in particular, is strictly less than a positive payoff of the non-initiating bidder with the same type. Lemma 5 shows that the ordering reverses once we move from the case of common values to the case of private values. The intuition behind this result is that all else equal, the non-initiating bidder in common value auctions bids higher than in private value auctions: not only does it simply update its bid in the face of stronger competition (recognizing that the initiating bidder has high valuation), which is the feature of private value auctions, but also it updates its valuation upwards leading to an even stronger incentive to bid higher. This completely erodes revenues of the boundary type of the initiating bidder in common value auctions but not in private value auctions. Moreover, this result means that information revelation by initiation, which was the driving force of the unraveling result in the common-values model, does not lower incentives of bidders to approach the seller in the private-values model. On the contrary, it helps: observing that no other bidder has approached the seller yet reveals information to the bidder, who contemplates approaching the seller, that valuations of other bidders are not too high.

#### 4.1.2 A seller-initiated auction

Consider an auction initiated by the seller. If the auction is initiated by the seller, none of the bidders has valuations above the initiation threshold  $\hat{s}$ , so all valuations are distributed i.i.d. over  $[0, \hat{s}]$ , where  $\hat{s}$  is some cut-off type. In this case, conditional on the auction

taking place, types of both bidders are drawn from the same distribution, so the analysis is standard (e.g., Krishna, 2010). Denote the equilibrium bids by  $\beta(s, \hat{s})$  and the inverse bidding function by  $\gamma(b, \hat{s})$ . The expected payoff of a bidder with signal  $s \in [0, \hat{s}]$  is

$$\Pi(b, s, \hat{s}) = \left( \frac{\gamma(b, \hat{s})}{\hat{s}} \right)^{N-1} (v(s) - b).$$

The first-order condition is

$$(N-1) \frac{\partial \gamma(b, \hat{s})}{\partial b} (v(s) - b) = \gamma(b, \hat{s}).$$

In equilibrium, the optimal bid is  $b = \beta(s, \hat{s})$ . Equivalently,  $s = \gamma(b, \hat{s})$ . Thus,

$$\frac{\partial \gamma(b, \hat{s})}{\partial b} = \frac{\gamma(b, \hat{s})}{(N-1)(v(\gamma(b, \hat{s})) - b)}. \quad (29)$$

This equation is to be solved subject to the boundary condition of  $\gamma(v(0), \hat{s}) = 0$ . Intuitively, this condition means that the lowest type,  $s = 0$ , bids his valuation,  $v(0)$ . In particular, note that because the initial value condition is independent of  $\hat{s}$ , and  $\hat{s}$  enters (29) only through  $\gamma(\cdot, \cdot)$ , the solution to (29) for a given  $b$  does not depend on  $\hat{s}$ . Thus, we can write the inverse bidding function as a function of the bid only,  $\gamma(b, \hat{s}) = \gamma(b)$ . The following lemma summarizes the equilibrium.

**Lemma 6 (equilibrium in the seller-initiated PV auction).** *There exists a unique equilibrium in the seller-initiated auction, when types of both bidders are below  $\hat{s}$ . The equilibrium inverse bidding strategy  $\gamma(b)$  solves (29) with the initial value condition  $\gamma(v(0)) = 0$ .*

Using an intuition similar to Lemma 5, the next lemma shows that a seller-initiated auction leads to a higher expected payoff to type  $\hat{s}$  than an auction initiated by her:

**Lemma 7.** *For any  $\hat{s}$ ,  $\Pi(\hat{s}, \hat{s}) \geq \Pi_I(\hat{s}, \hat{s})$ . The inequality is strict if  $\hat{s} > 0$ .*

**Proof.** Equilibrium bids take values from  $v(0)$  to  $\beta(\hat{s})$ . Hence,

$$\begin{aligned}
\Pi(\hat{s}, \hat{s}) &= \Pi(\beta(\hat{s}), \hat{s}, \hat{s}) \\
&\geq \Pi(\underline{b}(\hat{s}), \hat{s}, \hat{s}) \\
&= \left( \frac{\gamma(\underline{b}(\hat{s}), \hat{s})}{\hat{s}} \right)^{N-1} (v(\hat{s}) - \underline{b}(\hat{s})) \\
&\geq \left( \frac{v^{-1}(\underline{b}(\hat{s}))}{\hat{s}} \right)^{N-1} (v(s) - b) = \Pi_I(\hat{s}, \hat{s}),
\end{aligned}$$

with the strict inequality if  $\gamma(\underline{b}(\hat{s}), \hat{s}) > v^{-1}(\underline{b}(\hat{s}))$ . The last inequality follows from  $\beta(s, \hat{s}) \leq v(s)$ , which implies  $\gamma(b, \hat{s}) \geq v^{-1}(b)$ . Intuitively, no bidder bids above his valuation. This implies that if a bidder bids  $b$ , then his signal is at least  $v^{-1}(b)$ .

The comparison of Lemmas 5 and 7 implies that incentives of a bidder to approach the seller depend on whether her outside option is to wait for another bidder to approach the seller or for the seller to put the asset for sale himself. In the latter case, when a bidder expects the seller to sell itself soon, regardless of being contacted by bidders, a bidder benefits from waiting. In the extreme case, when a bidder expects the seller to sell the asset a second later, no type of the bidder approaches the seller, as  $\Pi(\hat{s}, \hat{s}) > \Pi_I(\hat{s}, \hat{s})$  for any  $\hat{s} > 0$ . Intuitively, information revealed by initiation hurts the initiating bidder, because non-initiating bidders realize that they face a stronger competitor and bid accordingly. This effect is absent in the seller-initiated auction. In the former case, when the outside option for a bidder is to wait until another bidder approaches the seller, information revelation does not create the same incentives for a bidder to wait.

#### 4.1.3 Off-equilibrium payoffs

Finally, for the next section we will also need off-equilibrium auction payoffs. First, consider type  $s < \hat{s}$  initiating the auction. The payoff of this type depends on the non-serious bids that types  $s < v^{-1}(\underline{b})$  of non-initiating bidders submit. The reason is that the non-serious bids of the static auction may become serious off-equilibrium in a dynamic setting, if a low type  $s < \hat{s}$  deviates and approaches the target. As long as non-serious bids are neither too low nor exceed valuations of bidders, any structure of non-serious bids can be an equilibrium outcome. For concreteness, we will assume that non-serious bidders bid their valuations.

Then, the payoff of type  $s < \hat{s}$  initiating the auction is

$$\Pi_I(s, \hat{s}) = \max_b \left( \frac{v^{-1}(b)}{\hat{s}} \right)^{N-1} (v(s) - b)$$

Such non-serious bids maximally lower the payoff that a bidder gets from initiation. If non-serious bids are lower, more types will approach the seller in dynamic equilibrium.

Second, consider type  $s > \hat{s}$  waiting until the auction is initiated by another bidder. He expects the initiating bidder to bid  $\beta_I(s)$ ,  $s \in [\hat{s}, 1]$  and  $N - 2$  of the other non-initiating bidders to bid  $\beta_N(s)$ ,  $s \in [0, \hat{s}]$ . His payoff from bidding  $b$  is  $\Pi_N(b, s, \hat{s})$ . Because  $\bar{b}(\hat{s}) = \arg \max_b \Pi_N(b, \hat{s}, \hat{s})$  and  $\gamma_I(\bar{b}(\hat{s}), \hat{s}) = \gamma_N(\bar{b}(\hat{s}), \hat{s}) = 1$ ,  $\Pi_N(b, s, \hat{s})$  is also maximized at  $\bar{b}(\hat{s})$  for any  $s > \hat{s}$ . Therefore,

$$\Pi_N(s, \hat{s}) = v(s) - \bar{b}(\hat{s})$$

for any  $s \geq \hat{s}$ .

Finally, consider type  $s > \hat{s}$  waiting until the auction is initiated by the seller. By the same logic,  $\bar{\beta}(\hat{s}) = \arg \max_b \Pi(b, s, \hat{s})$  for any  $s > \hat{s}$ . Therefore,

$$\Pi(s, \hat{s}) = v(s) - \bar{\beta}(\hat{s}).$$

Note that  $\Pi(s, \hat{s}) \geq \Pi_N(s, \hat{s})$  for  $s > \hat{s}$ , as the lemmas above show that  $\Pi(\hat{s}, \hat{s}) \geq \Pi_N(\hat{s}, \hat{s})$  and  $\Pi(s, \hat{s}) - \Pi_N(s, \hat{s}) = \bar{b}(\hat{s}) - \bar{\beta}(\hat{s}) = \Pi(\hat{s}, \hat{s}) - \Pi_N(\hat{s}, \hat{s})$ .

## 4.2 The Initiation Game: Stationary Solution

For this section, we will also focus on the stationary case, in which the cut-off is constant over time at some level  $\hat{s}$ . Our focus here is on the levels of  $\hat{s}$  that can be supported in the stationary solution. In the next section, we present a preliminary analysis of complete dynamics.

We first solve a bidder's problem taking the initiation strategy of the seller and all other bidders as given. Applying the symmetry condition, we will obtain the equilibrium initiation strategy of all bidders for any given constant initiation strategy of the seller. Then, we will solve the seller's problem taking the equilibrium strategy of bidders as given.



### 4.2.1 A bidder's problem

Suppose that the seller initiates the auction with probability  $\mu dt$  over any short interval of time  $[t, t + dt]$ , i.e.,  $S_t = \mu dt$ . We will solve for the symmetric equilibrium initiation strategy of bidders given  $\mu$  and later characterize the optimal strategy of the seller. Suppose that a bidder believes that each of  $N - 1$  other bidders approaches the seller if and only if their signal exceeds  $\hat{s}$ . Consider the remaining bidder with signal  $s$ . Let  $V(s, \hat{s}, \mu)$  denote the expected value of this bidder if his type is  $s$  and other bidders follow the cut-off rule  $\hat{s}$  to approach the seller.  $V(s, \hat{s}, \mu)$  satisfies

$$V(s, \hat{s}, \mu) = \max \left\{ \Pi_I(s, \hat{s}), \frac{(N-1)\lambda(1-\hat{s})\Pi_N(s, \hat{s}) + \mu\Pi(s, \hat{s}) + \lambda X}{r + (N-1)\lambda(1-\hat{s}) + \lambda + \mu} \right\} \quad (30)$$

Here,  $X$  denotes the value that a bidder gets when she is hit by the shock. If the shock is such that the bidder leaves the game,  $X$  is exogenous. If the shock is such that a bidder draws a new valuation, then  $X = \int_0^1 V(s', \hat{s}, \mu) ds'$ . The intuition behind (30) is as follows. The expected value to the bidder is the maximum between approaching the seller immediately and waiting. Approaching the seller immediately yields the expected value of  $\Pi_I(s, \hat{s})$ . Waiting yields the expected value that equals the second term of (30). With intensity  $(N-1)\lambda(1-\hat{s})$ , a rival bidder with type above  $\hat{s}$  appears and approaches the seller. In this case, the auction gets initiated by a rival bidder, so the bidder under consideration gets the expected value of  $\Pi_N(s, \hat{s})$ . With intensity  $\mu$ , the seller puts the asset for sale without waiting for any bidder to come. Finally, with intensity  $\lambda$ , the bidder gets a shock and obtains  $X$ . Because all events are independent, the expected value of waiting is given by the second term of (30).

By continuity of  $\Pi_I(\cdot)$ ,  $\Pi_N(\cdot)$ , and  $\Pi(\cdot)$ , the cut-off type must satisfy

$$\Pi_I(\hat{s}, \hat{s}) = \frac{(N-1)\lambda(1-\hat{s})\Pi_N(\hat{s}, \hat{s}) + \mu\Pi(\hat{s}, \hat{s}) + \lambda X}{r + (N-1)\lambda(1-\hat{s}) + \lambda + \mu}.$$

It is more convenient to re-write this as

$$r\Pi_I(\hat{s}, \hat{s}) + (N-1)\lambda(1-\hat{s})(\Pi_I(\hat{s}, \hat{s}) - \Pi_N(\hat{s}, \hat{s})) = \mu(\Pi(\hat{s}, \hat{s}) - \Pi_I(\hat{s}, \hat{s})) + \lambda(X - \Pi_I(\hat{s}, \hat{s})) \quad (31)$$

This equation has an intuitive interpretation. It states that for the indifferent type  $\hat{s}$ , the cost of waiting equals the benefit. The cost of waiting (the left-hand side) consists of two

components: delay in the surplus realized from the auction and the possibility that a rival bidder of the high type appears. The benefit of waiting is that the bidder delays expecting the seller to initiate the auction, as this leads to a higher expected payoff for the bidder. In addition, the bidder may get a shock and obtain the payoff of  $X$  upon it. This term is positive if the valuation of the bidder is low and negative if the valuation of the bidder is high.

Note that  $\Pi_N(\hat{s}, \hat{s}) = v(\hat{s}) - \bar{b}(\hat{s})$ ,  $\Pi(\hat{s}, \hat{s}) = v(\hat{s}) - \beta(\hat{s})$ , and  $\Pi_I(\hat{s}, \hat{s}) = \max_b \left( \frac{v^{-1}(b)}{\hat{s}} \right)^{N-1} (v(\hat{s}) - b)$ . Thus equation (31) takes a simple form. Figure 3, Panel A illustrates the typical behavior of costs and benefits of waiting as a function of  $\hat{s}$  and the equilibrium threshold for the realistic parametrization:  $N = 2$ ,  $v(s) = s$ , and  $\hat{s} = 0.5$ ,  $r = 0.05$ ,  $\lambda = 0.5$  (bidders change type on average every two years),  $\mu = 0.2$  (sellers put their asset for sale on average after five years),  $X = 0.15$ . Here, a single solution obtains. Panel B illustrates the case of two solutions, the larger of which is unstable, for the following parametrization:  $N = 2$ ,  $v(s) = s$ , and  $\hat{s} = 0.5$ ,  $r = 0.05$ ,  $\lambda = 0.5$ ,  $\mu = 0.55$ ,  $X = 0.03$ .

The following proposition presents some analysis of (31):

**Proposition 2.** *Let  $\hat{s}$  be the equilibrium cut-off type. It has the following properties:*

1.  $\hat{s} > 0$ ;
2. If  $X < \bar{X}$ , where

$$\bar{X} = \left(1 + \frac{r}{\lambda}\right) \Pi_I(1, 1) - \frac{\mu}{\lambda} (\Pi(1, 1) - \Pi_I(1, 1)),$$

then  $\hat{s} < 1$ ;

3. If the equilibrium cut-off type  $\hat{s}$  for a given  $\mu$ , denoted  $\hat{s}(\mu)$ , is unique, then  $\hat{s}(\mu)$  is always increasing in  $\mu$  and strictly increasing if  $\hat{s}(\mu) < 1$ .
4. There exists a finite  $\mu^*$ , such that for any  $\mu > \mu^*$ , the unique equilibrium cut-off type is  $\hat{s} = 1$ .

**Proof.** Part (1) of the proposition follows from  $\Pi_I(0, 0) = \Pi_N(0, 0) = \Pi(0, 0) = 0$ . Then, equation (31) implies  $0 = \lambda X$ , which cannot hold because  $X > 0$ . Consider part (2) of the proposition. For  $\hat{s} = 1$ , (30) implies that approaching the seller yields  $\Pi_I(1, 1)$ ,

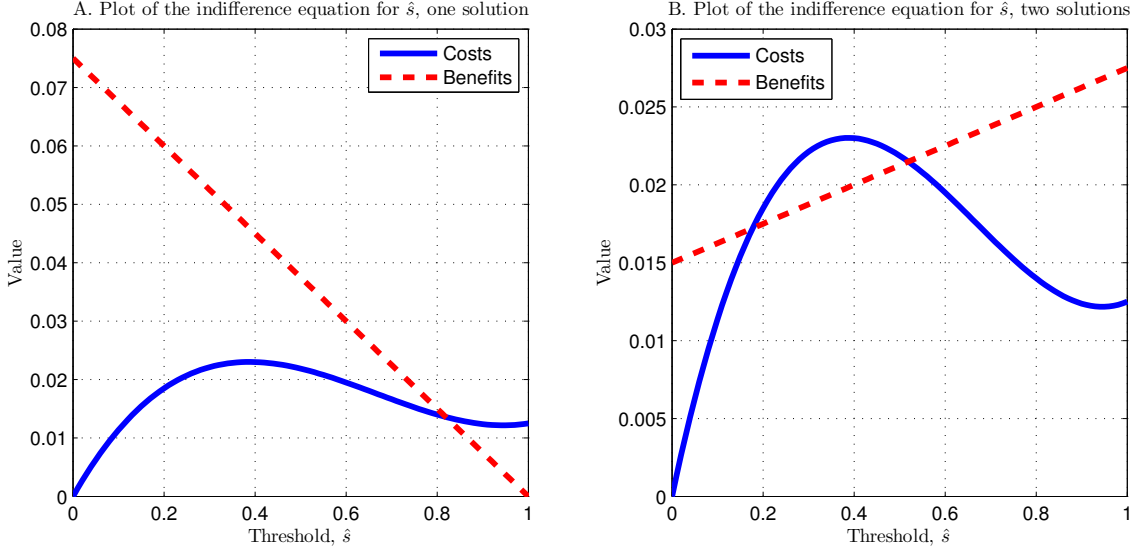


Figure 3: The behavior of costs and benefits of waiting as a function of  $\hat{s}$  and the equilibrium  $\hat{s}$ .

while waiting yields  $\frac{\mu\Pi(1,1)+\lambda X}{r+\lambda+\mu}$ . If  $X < \bar{X}$ , then approaching the seller strictly dominates waiting. Thus, types  $s$  close enough to  $\hat{s} = 1$  strictly benefit from approaching the seller, so  $\hat{s} = 1$  is not an equilibrium. Next, consider part (3) of the proposition. If (31) is solved by a unique  $\hat{s}$ , then function

$$\begin{aligned} f_{\mu}(s) \equiv & r\Pi_I(s, s) + (N-1)\lambda(1-s)(\Pi_I(s, s) - \Pi_N(s, s)) \\ & - \mu(\Pi(s, s) - \Pi_I(s, s)) - \lambda(X - \Pi_I(s, s)) \end{aligned}$$

crosses zero at  $s = \hat{s}$  from below. This follows from continuity of  $f_{\mu}(s)$  and  $f_{\mu}(0) = -\lambda X < 0$ . Next,

$$f_{\mu'}(s) - f_{\mu}(s) = -(\mu' - \mu)(\Pi(s, s) - \Pi_I(s, s)) < 0,$$

for all  $s > 0$  and  $\mu' > \mu$ . Because  $\hat{s} > 0$ , if  $f_{\mu}(\hat{s}) = 0$ , then  $f_{\mu'}(\hat{s}) < 0$ . Uniqueness of the solution means that  $\hat{s}(\mu') > \hat{s}(\mu)$ . Finally, consider part (4) of the proposition. Because all  $\Pi_I(\cdot)$ ,  $\Pi_N(\cdot)$ , and  $\Pi_I(\cdot)$  are finite, and  $\Pi(s, s) - \Pi_I(s, s) > 0$ ,  $\lim_{\mu \rightarrow \infty} f_{\mu}(s) = -\infty$  for all  $s > 0$ . Therefore, waiting strictly dominates approaching the seller for any  $\hat{s}$  for a high enough  $\mu$ .

Figure 4 illustrates the behavior of the equilibrium threshold,  $\hat{s}(\mu)$ , for the case  $N = 2$ ,

$v(s) = s$ , and  $\hat{s} = 0.5$ ,  $r = 0.05$ ,  $\lambda = 0.5$ ,  $X = 0.15$ .

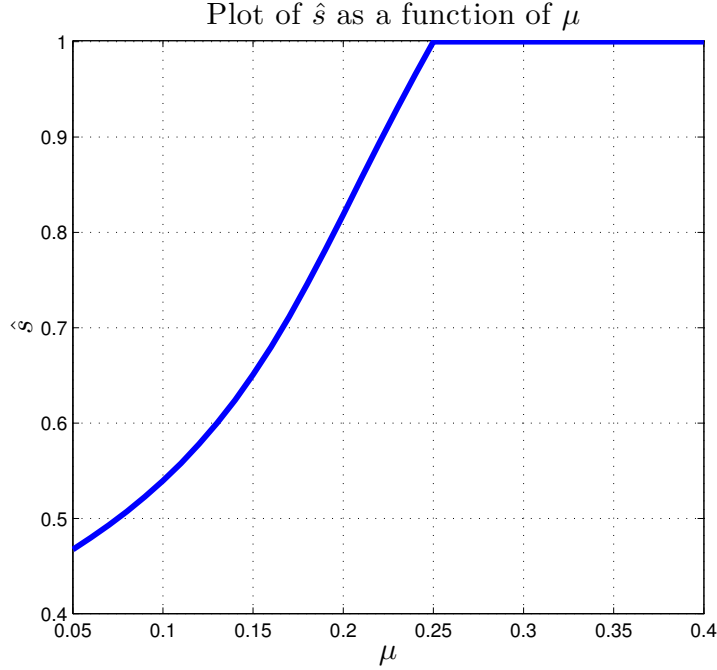


Figure 4: The behavior of the equilibrium threshold,  $\hat{s}(\mu)$ , as a function of  $\mu$ .

#### 4.2.2 The seller's problem

Given  $\hat{s}$ , consider the best response of the seller. If the seller waits until he is approached by bidders, his payoff is

$$\frac{N\lambda(1-\hat{s})}{r+N\lambda(1-\hat{s})}R_B(\hat{s}),$$

where  $R_B(\hat{s})$  are expected revenues in the bidder-initiated auction, when types  $\hat{s}$  and above approach the seller initiating the auction. If the seller deviates and initiates the auction himself, his payoff is  $R(\hat{s})$ , where  $R_S(\hat{s})$  are expected revenues in the standard symmetric auction when types of bidders are distributed over  $[0, \hat{s}]$ . Thus, the seller has no incentives to deviate if and only if

$$\frac{N\lambda(1-\hat{s})}{r+N\lambda(1-\hat{s})}R_B(\hat{s}) \geq R_S(\hat{s}). \quad (32)$$

If this condition is satisfied, then there exists an equilibrium in which all auctions are bidder-initiated. Specifically, the seller waits until he is approached by a type  $s > \hat{s}$ .

### 4.2.3 Equilibria

Combining the derivations of the previous two sections, we can characterize the set of cut-off types  $\hat{s}$  that can be consistent with the stationary outcome of the initiation game.

**Proposition 3 (equilibrium with only seller-initiated auctions).** *There always exists an equilibrium in which  $\hat{s} = 1$  and  $\mu = \infty$ .*

**Proof.** First, consider the decision of the seller to deviate. For  $\hat{s} = 1$ , condition (32) is violated, because the left-hand side of (32) equals zero and the right-hand side equals  $R_S(1) > 0$ . Thus, the seller does not benefit from deviation. Second, consider the decision of a bidder to deviate. As Proposition 1 shows, for any  $\mu > \mu^*$ , in particular, for  $\mu = \infty$ ,  $\hat{s} = 1$  is the unique equilibrium initiation strategy of bidders for a fixed  $\mu$ . Thus, no bidder benefits from deviation.

Proposition 3 implies that there always exists an equilibrium in which bidders never approach the seller, and the seller puts the asset for sale immediately. Intuitively, if the seller believes that no bidder ever comes to the seller, then delaying the sale hurts the seller, as delay does not allow to screen types of bidders. Similarly, if bidders believe that the seller will put the asset for sale soon, no bidder benefits from approaching the seller.

To have both seller- and bidder-initiated auctions in equilibrium at the same time, the seller must play mixed strategies. To play mixed strategies, the seller must be indifferent between initiating the auction himself and waiting until he is approached by a bidder:

$$\frac{N\lambda(1-\hat{s})}{r+N\lambda(1-\hat{s})}R_B(\hat{s}) = R_S(\hat{s}). \quad (33)$$

This leads to the following result:

**Proposition 4 (equilibria with seller- and bidder-initiated auctions).** *Consider a pair  $(\mu, \hat{s})$  with  $\mu > 0$ . If it is an equilibrium,  $\hat{s}$  and  $\mu$  satisfy (33) and (31).*

Note that equation (33) is independent of  $\mu$ . Thus, it determines  $\hat{s}$ . Given the value of  $\hat{s}$ , equation (31) determines the corresponding frequency  $\mu$  with which the seller initiates the auction without being approached by a bidder, so that the cut-off type is exactly indifferent

between waiting and approaching the seller:

$$\mu = \frac{(r + \lambda) \Pi_I(\hat{s}, \hat{s}) + (N - 1) \lambda (1 - \hat{s}) (\Pi_I(\hat{s}, \hat{s}) - \Pi_N(\hat{s}, \hat{s})) - \lambda X}{\Pi(\hat{s}, \hat{s}) - \Pi_I(\hat{s}, \hat{s})}. \quad (34)$$

Finally, there can be an equilibrium in which all auctions are bidder-initiated:

**Proposition 5 (equilibrium with only bidder-initiated auctions).** *The necessary condition for an equilibrium with only bidder-initiated auction is that the cut-off type  $\hat{s}$  satisfies*

$$r \Pi_I(\hat{s}, \hat{s}) + (N - 1) \lambda (1 - \hat{s}) (\Pi_I(\hat{s}, \hat{s}) - \Pi_N(\hat{s}, \hat{s})) = \lambda (X - \Pi_I(\hat{s}, \hat{s})), \quad (35)$$

$$\frac{N \lambda (1 - \hat{s})}{r + N \lambda (1 - \hat{s})} R_B(\hat{s}) \geq R_S(\hat{s}). \quad (36)$$

These conditions are intuitive. The first condition is the special case of (31) for  $\mu = 0$ . The second condition is the incentive-compatibility condition for the seller, which states that waiting for the bidder to approach the seller generates at least the same expected payoff as immediate initiation.

Figure 5 illustrates equilibria with seller- and bidder-initiated auctions (three in total) for the case  $N = 2$ ,  $v(s) = s$ ,  $r = 0.05$ ,  $\lambda = 0.5$ ,  $X = 0.15$ .

### 4.3 The Initiation Game: Complete Dynamics

The stationary restriction of Section 4.2 does not say anything about whether and how the initiating game among bidders and the seller reaches stationarity. In this section, we expand our analysis to a non-stationary setting: we assume that at time  $t = 0$ , the bidders receive private signals which are distributed uniformly over  $[0, 1]$ . In other words, we remove restriction 2. of Section 4.2. Instead, we conjecture and later confirm that the cut-off  $\hat{s}(t)$  is a decreasing function of time:  $\hat{s}'(t) < 0$ . Now, at any time a bidder can initiate the auction either because his identity (or type) changes or because the decreasing initiation threshold reaches his current type: both of these events can happen with positive probability and, from the perspective of non-initiating bidders, are indistinguishable. The bidders will adjust their strategies, both at the bidding and initiation stage, in response

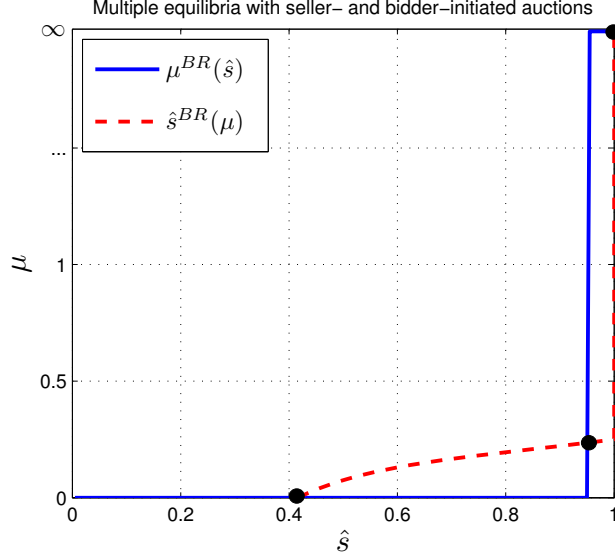


Figure 5: **Equilibria with seller- and bidder-initiated auctions.**

to this additional uncertainty. As a result, we start to explore complete dynamics with solving for the equilibria in bidder- and seller-initiated auctions off-steady state using our conjecture about  $\hat{s}(t)$ . Next, we solve a bidder's problem taking the initiation strategy of the seller and all other bidders as given, and confirm the conjecture. At the moment, we assume that the seller can initiate the auction with constant probability  $\mu dt$  over any short interval of time  $dt$ . To be completed: solve for the joint equilibrium initiation strategies of all bidders and the seller.

#### 4.3.1 Equilibrium in a bidder-initiated auction

Suppose that bidder 1 approaches the seller if and only if his signal exceeds  $\hat{s}(t)$  which in this subsection, for simplicity, we will still denote as  $\hat{s}$ . At the auction stage bidder 1 believes that the types of other bidders are distributed uniformly over  $[0, \hat{s}]$ . However, because  $\hat{s}$  is, by conjecture, decreasing with time, other bidders believe that the type of bidder 1 is either equal exactly  $\hat{s}$  with conditional probability  $p = \frac{-\hat{s}'/\hat{s}}{\lambda(1-\hat{s})-\hat{s}'/\hat{s}}$  or is distributed uniformly over  $[\hat{s}, 1]$  with conditional probability  $1 - p$ .  $p$  accounts for the fact that the probability over any short interval of time  $dt$  that the initiating bidder's identity or type has changed to a value in the interval  $[\hat{s}, 1]$  is  $\lambda dt(1 - \hat{s})$  while the same probability that the initiating bidder's type remains the same but initiation is triggered by a change in  $\hat{s}$

is  $-\hat{s}'dt/\hat{s}$ . Unlike before, we will define equilibrium bidding functions through cumulative distribution functions. Specifically, let  $F_N(b, \hat{s}, p)$  denote the probability that bid  $b$  wins against a random bid of a single rival non-initiating bidder.<sup>10</sup> Similarly, let  $F_I(b, \hat{s}, p)$  denote the probability that bid  $b$  wins against the random bid of the rival initiating bidder.

We conjecture that the equilibrium takes the following form. The set of “serious” bids is  $[\underline{b}(\hat{s}, p), \bar{b}(\hat{s}, p)]$ . The non-initiating bidder  $i$  bids  $\beta_N(s_i, \hat{s}, p) \in [\underline{b}(\hat{s}, p), \bar{b}(\hat{s}, p)]$ , if  $s_i \geq \underline{b}(\hat{s}, p)$ . The initiating bidder of type  $\hat{s}$  plays the mixed strategy of bidding over interval  $[\underline{b}(\hat{s}, p), \hat{b}(\hat{s}, p)]$  for some  $\hat{b}(\hat{s}, p) \in [\underline{b}(\hat{s}, p), \bar{b}(\hat{s}, p)]$ . The initiating bidder of type  $s_i > \hat{s}$  bids  $\beta_I(s_i, \hat{s}, p) \in [\hat{b}(\hat{s}, p), \bar{b}(\hat{s}, p)]$ . Note that  $F_I(\hat{b}) = p$ .

Consider the non-initiating bidder with type  $s$ . He solves the problem

$$\max_b F_I(b, \hat{s}, p) F_N^{N-2}(b, \hat{s}, p) (v(s) - b).$$

Intuitively, if a non-initiating bidder bids  $b$ , it wins against the bid of the initiating bidder with probability  $F_I(b, \hat{s}, p)$  and against the bid of each single rival non-initiating bidder with probability  $F_N(b, \hat{s}, p)$ . Conditional on winning against bids of every rival bidder, the non-initiating bidder gets the payoff of  $v(s) - b$ . In separating equilibrium, a given bid  $b \in [\underline{b}(\hat{s}, p), \bar{b}(\hat{s}, p)]$  is an optimal bid for type  $s(b)$  such that  $s(b)/\hat{s} = F_N(b, \hat{s}, p)$ , that is, the chance of winning with bid  $b$  against a non-initiating bidder is  $\text{Prob}(s < s(b)|s \in U[0, \hat{s}]) = s(b)/\hat{s}$ . Therefore, the following first-order condition must hold:

$$\left( \frac{\partial F_I(b, \hat{s}, p)}{\partial b} + (N-2)F_I(b, \hat{s}, p) \frac{\frac{\partial F_N(b, \hat{s}, p)}{\partial b}}{F_N(b, \hat{s}, p)} \right) (v(F_N(b, \hat{s}, p)\hat{s}) - b) = F_I(b, \hat{s}, p). \quad (37)$$

Next, consider the initiating bidder of type  $\hat{s}$ . Randomization among bids  $b \in [\underline{b}(\hat{s}, p), \hat{b}(\hat{s}, p)]$  requires that

$$F_N^{N-1}(b, \hat{s}, p) (v(\hat{s}) - b) = C \quad (38)$$

for any  $b \in [\underline{b}(\hat{s}, p), \hat{b}(\hat{s}, p)]$  and some constant  $C$ . If the above condition does not hold the initiating bidder will prefer to deviate to the most profitable set of bids.

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<sup>10</sup>The bid of a rival non-initiating bidder is random from the perspective of the bidder making bid  $b$  because the latter does not know the signal of the former.



Finally, consider the initiating bidder of type  $s > \hat{s}$ . He solves the problem

$$\max_b F_N^{N-1}(b, \hat{s}, p) (v(s) - b),$$

which yields the first-order condition  $(N-1) \frac{\partial F_N(b, \hat{s}, p)}{\partial b} (v(s) - b) = F_N(b, \hat{s}, p)$ . In separating equilibrium, a given bid  $b \in [\hat{b}(\hat{s}, p), \bar{b}(\hat{s}, p)]$  is an optimal bid for type  $s(b)$  such that  $p + (1-p) \frac{s(b) - \hat{s}}{1 - \hat{s}} = F_I(b, \hat{s}, p)$ , that is, the chance of winning with bid  $b \geq \hat{b}(\hat{s}, p)$  against the initiating bidder is  $p \text{Prob}(s < s(b) | s = \hat{s}) + (1-p) \text{Prob}(s < s(b) | s \in U[\hat{s}, 1]) = p + (1-p) \frac{s(b) - \hat{s}}{1 - \hat{s}}$ . Therefore, we have the following optimal condition:

$$(N-1) \frac{\partial F_N(b, \hat{s}, p)}{\partial b} \left( v \left( \hat{s} + \frac{F_I(b, \hat{s}, p) - p}{1-p} (1 - \hat{s}) \right) - b \right) = F_N(b, \hat{s}, p), \quad (39)$$

for any  $b \in [\hat{b}(\hat{s}, p), \bar{b}(\hat{s}, p)]$ .

The system of three equations, (37)–(39), is solved subject to the following boundary conditions:

$$F_I(\bar{b}(\hat{s}, p), \hat{s}, p) = F_N(\bar{b}(\hat{s}, p), \hat{s}, p) = 1, \quad (40)$$

$$F_I(\underline{b}(\hat{s}, p), \hat{s}, p) = 0, \quad (41)$$

$$F_N(\underline{b}(\hat{s}, p), \hat{s}, p) = \frac{v^{-1}(\underline{b}(\hat{s}, p))}{\hat{s}}, \quad (42)$$

$$F_I(\hat{b}(\hat{s}, p), \hat{s}, p) = p. \quad (43)$$

These conditions are similar to (23)–(26) but also account for randomization by the boundary type of the initiating bidder. From the boundary condition (42) and (38), we can write

$$F_N(b, \hat{s}, p) = \frac{v^{-1}(\underline{b}(\hat{s}, p))}{\hat{s}} \left( \frac{v(\hat{s}) - \underline{b}(\hat{s}, p)}{v(\hat{s}) - b} \right)^{\frac{1}{N-1}} \quad (44)$$

for any  $b \in [\underline{b}(\hat{s}, p), \hat{b}(\hat{s}, p)]$ .

The minimum bid  $\underline{b}(\hat{s}, p)$  is determined from the following argument. Bid  $\underline{b}(\hat{s}, p)$  must be optimal for the initiating bidder of type  $\hat{s}$ . Therefore,

$$\left( \frac{v^{-1}(\underline{b}(\hat{s}, p))}{\hat{s}} \right)^{N-1} (v(\hat{s}) - \underline{b}) \geq F_N^{N-1}(b, \hat{s}, p) (v(\hat{s}) - b) \quad \forall b.$$

In equilibrium, no type of the initiating bidder ever bids above her valuation:  $\beta_N(s) \leq v(s)$ . Therefore,  $\beta_N^{-1}(b) \geq v^{-1}(b)$ , which implies  $F_N(b) \geq \frac{v^{-1}(b)}{\hat{s}}$ . Hence,

$$\left( \frac{v^{-1}(\underline{b}(\hat{s}, p))}{\hat{s}} \right)^{N-1} (v(\hat{s}) - \underline{b}(\hat{s}, p)) \geq \left( \frac{v^{-1}(b)}{\hat{s}} \right)^{N-1} (v(\hat{s}) - b) \quad \forall b.$$

Therefore,

$$\underline{b}(\hat{s}, p) = \arg \max_b \left( \frac{v^{-1}(b)}{\hat{s}} \right)^{N-1} (v(\hat{s}) - b), \quad (45)$$

which yields

$$(N-1) \frac{v(\hat{s}) - \underline{b}(\hat{s}, p)}{v'(\underline{b}(\hat{s}, p))} = v^{-1}(\underline{b}(\hat{s}, p)). \quad (46)$$

The following lemma summarizes the unique equilibrium in the bidder-initiated first-price auction in the off-steady state setting:

**Lemma 8 (equilibrium in the bidder-initiated PV auction, off-steady state setting).** *There exists a unique (up to the non-serious bids of types  $s < v^{-1}(\underline{b}(\hat{s}, p))$  of non-initiating bidders) equilibrium in the bidder-initiated auction. The equilibrium probabilities of winning with bid  $b$  against a non-initiating bidder and the initiating bidder,  $F_N(b, \hat{s}, p)$  and  $F_I(b, \hat{s}, p)$ , satisfy equations (37)–(39) with boundary condition (40)–(43) and the lowest serious bid is given by (45).*

Section 5.3 illustrates the solution specializing to the case  $N = 2$  and  $v(s) = s$ . Figure 6 illustrates the equilibrium bidding strategies for the case  $N = 2$ ,  $v(s) = s$ ,  $\hat{s} = 0.5$ , and  $p = 0.5$ . For the same case, Figure 7 shows the behavior of  $\hat{b}(\hat{s}, p)$  and  $\bar{b}(\hat{s}, p)$  as functions of the conditional probability that the initiating bidder is of the boundary type,  $p$ . As  $p$  increases, the gap between the two threshold bids decreases. As can be seen from the graph,  $\hat{b}(\hat{s}, p)$  and  $\bar{b}(\hat{s}, p)$ , indeed, constitute an equilibrium not only in bidding strategies but also in the roles of the bidders: the boundary type of the initiating bidder, who receives the payoff of  $\hat{s}/4 = 1/4$ , does not find it profitable to become a non-initiating bidder and win with probability one: in case of such deviation, the payoff is  $\hat{s} - \bar{b}(\hat{s}, p) \leq 1/4$  and the equality obtains only for  $p = 1$ .

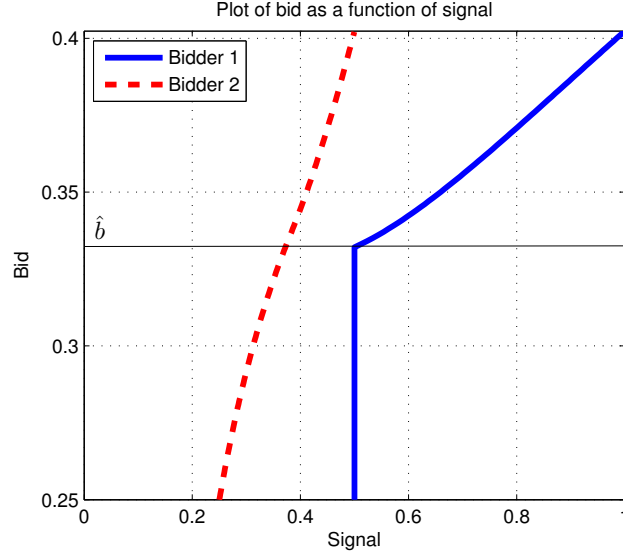


Figure 6: **Equilibrium bids in a bidder-initiated private-value auction, off-steady state setting.** The figure plots the equilibrium bids as functions of signals for the initiating bidder (the blue normal line) and the other bidder (the red dashed line) in the off-steady state setting.

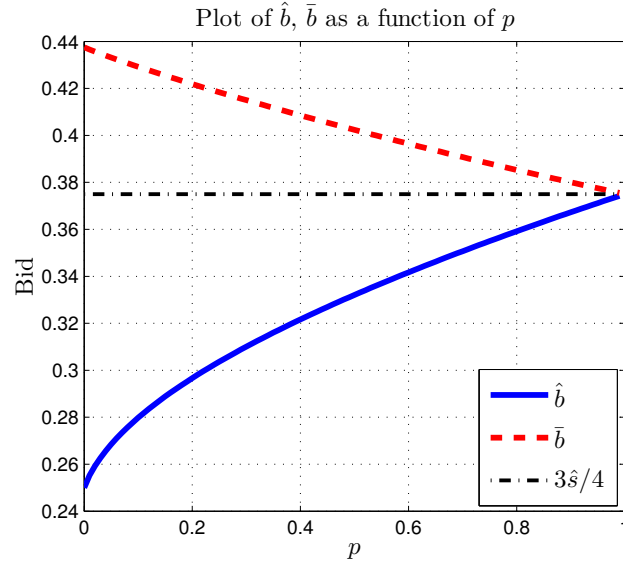


Figure 7: **Equilibrium threshold bids in a bidder-initiated private-value auction, off-steady state setting.** The figure plots the equilibrium threshold bids,  $\hat{b}$  and  $\bar{b}$ , as functions of the conditional probability that the initiating bidder is of the boundary type,  $p$ , in the off-steady state setting.

### 4.3.2 Equilibrium in a seller-initiated auction

If the auction is initiated by the seller, all valuations are distributed i.i.d. over  $[0, \hat{s}]$  where  $\hat{s}$  is time-varying cut-off type. As a result, auction outcomes are similar to Section 4.1.2. The following lemma summarizes the equilibrium:

**Lemma 9 (equilibrium in the seller-initiated PV auction, off-steady state setting).** *There exists a unique equilibrium in the seller-initiated auction, when types of both bidders are below  $\hat{s}$ . The equilibrium inverse bidding strategy  $\gamma(b)$  solves (29) with the initial value condition  $\gamma(v(0)) = 0$ .*

### 4.3.3 A bidder's initiation problem

As before, we focus on symmetric Markov perfect equilibria, in which bidders play cut-off strategies. In contrast to Section 4.2., we remove the stationarity restriction. We solve a bidder's problem taking the initiation strategy of the seller and all other bidders as given (currently, we assume that the seller initiates with the same probability over any short interval of time). Applying the symmetry condition, we will obtain the equilibrium initiation strategy of all bidders for any given constant initiation strategy of the seller.

Suppose that the seller initiates the auction with probability  $\mu dt$  over any short interval of time  $[t, t + dt]$ , i.e.,  $S_t = \mu dt$ . Suppose that a bidder believes that each of  $N - 1$  other bidders approaches the seller either because her signal strictly exceeds  $\hat{s}$ , which happens with probability  $\lambda(1 - \hat{s})dt$ , or because a decreasing  $\hat{s}$  reaches its current signal, which happens with probability  $-\hat{s}'dt/\hat{s}$ . Consider the remaining bidder with signal  $s$ . Let  $V(s, t, \mu)$  denote the expected value of this bidder if his type is  $s$  and other bidders follow the cut-off rule  $\hat{s}(t)$  to approach the seller.  $V(s, t, \mu)$  satisfies

$$\begin{aligned} V(s, t, \mu) &= \max \left\{ \Pi_I(s, \hat{s}, p), \frac{V'_t(s, t, \mu) + (N - 1)(\lambda(1 - \hat{s}) - \hat{s}'/\hat{s})\Pi_N(s, \hat{s}, p) + \mu\Pi(s, \hat{s}) + \lambda X}{r + (N - 1)(\lambda(1 - \hat{s}) - \hat{s}'/\hat{s}) + \lambda + \mu} \right\} \\ &= \max \left\{ \Pi_I(s, \hat{s}, p), \frac{V'_t(s, t, \mu) + (N - 1)\lambda\frac{1 - \hat{s}}{1 - p}\Pi_N(s, \hat{s}, p) + \mu\Pi(s, \hat{s}) + \lambda X}{r + (N - 1)\lambda\frac{1 - \hat{s}}{1 - p} + \lambda + \mu} \right\}. \end{aligned} \quad (47)$$

Here,  $X$  denotes the value that a bidder gets when she is hit by the shock. If the shock is such that the bidder leaves the game,  $X$  is exogenous. For the off-steady state setting, we

specialize to this assumption. The intuition behind (47) is as follows. The expected value to the bidder is the maximum between approaching the seller immediately and waiting. Approaching the bidder immediately yields the expected value of  $\Pi_I(s, \hat{s}, p)$ . Waiting yields the expected value that equals the second term of (47). The change in the continuation value with time is captured by  $V'_t(s, t, \mu)$ . With intensity  $(N-1)\lambda(1-\hat{s})$ , a rival bidder with type above  $\hat{s}$  appears and approaches the seller. Also, with intensity  $-(N-1)\hat{s}'/\hat{s} = (N-1)\frac{p}{1-p}\lambda(1-\hat{s})$ , one of the rival bidders whose type has not changed reaches the decreasing initiation threshold. In these cases, the auction gets initiated by a rival bidder, so the bidder under consideration gets the expected value of  $\Pi_N(s, \hat{s}, p)$ . With intensity  $\mu$ , the seller puts the asset for sale without waiting for any bidder to come. Finally, with intensity  $\lambda$ , the bidder gets a shock and obtains  $X$ . Because all events are independent, the expected value of waiting is given by the second term of (47).

By continuity of  $\Pi_I(\cdot)$ ,  $\Pi_N(\cdot)$ , and  $\Pi(\cdot)$ , the cut-off type must satisfy

$$\Pi_I(\hat{s}, \hat{s}, p) = \frac{V'_t(\hat{s}, t, \mu) + (N-1)\lambda\frac{1-\hat{s}}{1-p}\Pi_N(\hat{s}, \hat{s}, p) + \mu\Pi(\hat{s}, \hat{s}) + \lambda X}{r + (N-1)\lambda\frac{1-\hat{s}}{1-p} + \lambda + \mu}.$$

In addition, the smooth-pasting condition at  $\hat{s}$  must be satisfied:

$$V'_t(s, t, \mu) = \Pi'_{I,2}(\hat{s}, \hat{s}, p)\hat{s}' + \Pi'_{I,3}(\hat{s}, \hat{s}, p)p'.$$

**Lemma 10 (Smooth-pasting condition for the value function, off-steady state setting).** *The smooth-pasting condition for the boundary type of the initiating bidder,  $\hat{s}$ , is*

$$V'_t(s, t, \mu) = -\frac{p}{1-p}\lambda\hat{s}(1-\hat{s})(N-1)F_N^{N-2}(\hat{b}(\hat{s}, p), \hat{s}, p)\frac{\partial F_N(\hat{b}(\hat{s}, p), \hat{s}, p)}{\partial \hat{s}}(v(\hat{s}) - \hat{b}(\hat{s}, p)). \quad (48)$$

**Proof.** First, take a partial derivative of  $\Pi_I(s, \hat{s}, p)$  with respect to  $\hat{s}$  and calculate it at point  $s \rightarrow \hat{s}+$ , using that for the initiating bidder,  $\lim_{s \rightarrow \hat{s}+} b(s, \hat{s}, p) = \hat{b}(\hat{s}, p)$ :

$$\begin{aligned} \Pi'_{I,2}(\hat{s}, \hat{s}, p) &= (N-1)F_N^{N-2}(\hat{b}(\hat{s}, p), \hat{s}, p) \left( \frac{\partial F_N(\hat{b}(\hat{s}, p), \hat{s}, p)}{\partial \hat{b}} \frac{\partial \hat{b}(\hat{s}, p)}{\partial \hat{s}} + \frac{\partial F_N(\hat{b}(\hat{s}, p), \hat{s}, p)}{\partial \hat{s}} \right) (v(\hat{s}) - \hat{b}(\hat{s}, p)) \\ &\quad - F_N^{N-1}(\hat{b}(\hat{s}, p), \hat{s}, p) \frac{\partial \hat{b}(\hat{s}, p)}{\partial \hat{s}}. \end{aligned}$$

The first-order condition for the initiating bidder at  $s \rightarrow \hat{s}+$  is

$$(N-1) \frac{\partial F_N(\hat{b}(\hat{s}, p), \hat{s}, p)}{\partial \hat{b}} (v(\hat{s}) - b(\hat{s}, p)) = F_N(\hat{b}(\hat{s}, p), \hat{s}, p),$$

which gives

$$\Pi'_{I,2}(\hat{s}, \hat{s}, p) = (N-1) F_N^{N-2}(\hat{b}(\hat{s}, p), \hat{s}, p) \frac{\partial F_N(\hat{b}(\hat{s}, p), \hat{s}, p)}{\partial \hat{s}} (v(\hat{s}) - b(\hat{s}, p)).$$

From (44),  $F_N(\hat{b}(\hat{s}, p), \hat{s}, p) = \frac{v^{-1}(b(\hat{s}, p))}{\hat{s}} \left( \frac{v(\hat{s}) - b(\hat{s}, p)}{v(\hat{s}) - \hat{b}(\hat{s}, p)} \right)^{\frac{1}{N-1}}$  is fully characterized. Second, take a partial derivative of  $\Pi_I(s, \hat{s}, p)$  with respect to  $p$  and calculate it at point  $s \rightarrow \hat{s}+$ , using that for the initiating bidder,  $\lim_{s \rightarrow \hat{s}+} b(s, \hat{s}, p) = \hat{b}(\hat{s}, p)$ :

$$\begin{aligned} \Pi'_{I,3}(\hat{s}, \hat{s}, p) &= (N-1) F_N^{N-2}(\hat{b}(\hat{s}, p), \hat{s}, p) \left( \frac{\partial F_N(\hat{b}(\hat{s}, p), \hat{s}, p)}{\partial \hat{b}} \frac{\partial \hat{b}(\hat{s}, p)}{\partial p} + \frac{\partial F_N(\hat{b}(\hat{s}, p), \hat{s}, p)}{\partial p} \right) (v(\hat{s}) - b(\hat{s}, p)) \\ &\quad - F_N^{N-1}(\hat{b}(\hat{s}, p), \hat{s}, p) \frac{\partial \hat{b}(\hat{s}, p)}{\partial p}. \end{aligned}$$

Again, using the first-order condition for the initiating bidder at  $s \rightarrow \hat{s}+$ ,

$$\Pi'_{I,3}(\hat{s}, \hat{s}, p) = (N-1) F_N^{N-2}(\hat{b}(\hat{s}, p), \hat{s}, p) \frac{\partial F_N(\hat{b}(\hat{s}, p), \hat{s}, p)}{\partial p} (v(\hat{s}) - b(\hat{s}, p)).$$

Note that for a given  $b$ ,  $F_N(\hat{b}(\hat{s}, p), \hat{s}, p) = 0$  because it does not depend on  $p$ . Hence,

$$\Pi'_{I,3}(\hat{s}, \hat{s}, p) = 0.$$

Finally,  $\hat{s}' = -\frac{p}{1-p} \lambda \hat{s} (1 - \hat{s})$ . This concludes the proof. An example for  $N = 2$ ,  $v(s) = s$  is given in Section 5.4.

It is more convenient to re-write the equation for the cut-off type as

$$r \Pi_I(\hat{s}, \hat{s}, p) + (N-1) \lambda \frac{1 - \hat{s}}{1 - p} (\Pi_I(\hat{s}, \hat{s}, p) - \Pi_N(\hat{s}, \hat{s}, p)) - V'_t(\hat{s}, t, \mu) = \mu (\Pi(\hat{s}, \hat{s}) - \Pi_I(\hat{s}, \hat{s}, p)) + \lambda (X - \Pi_I(\hat{s}, \hat{s}, p)) \quad (49)$$

Equation (49) is a differential equation on  $\hat{s}(t)$  (because  $p$  is a function of  $\hat{s}'(t)$ ) with the

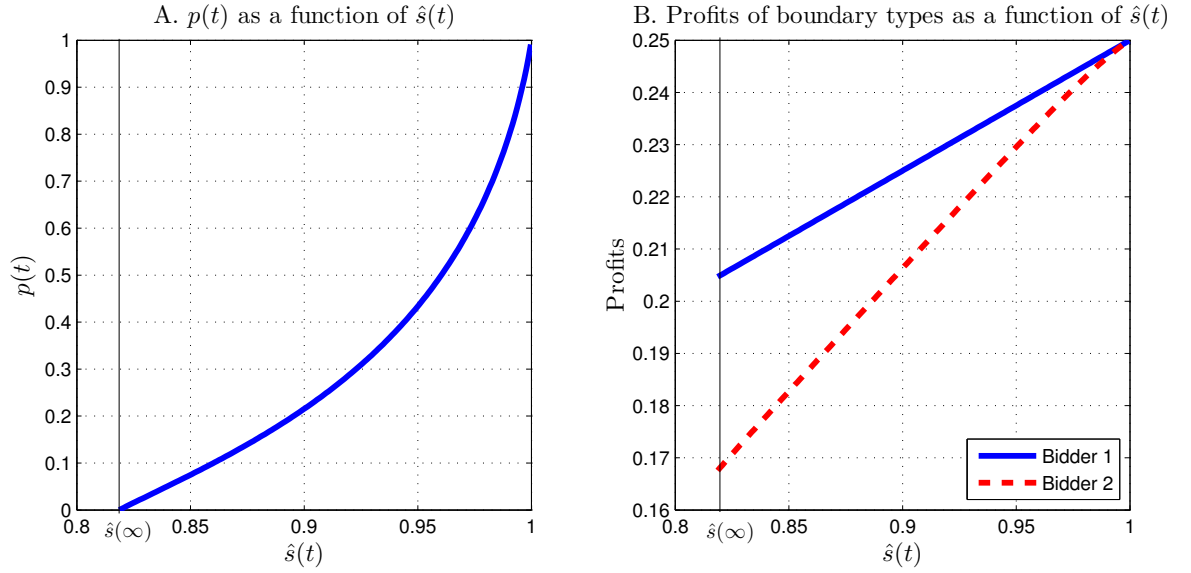


Figure 8: **The dynamics of the boundary type,  $\hat{s}$ , and profits of boundary-type bidders.** Panel A plots the behavior of the conditional probability that the initiating bidder is of the boundary type,  $p(t)$ , as a function of the boundary type,  $\hat{s}(t)$ . The dynamics of the boundary type is given by the differential equation  $\hat{s}'(t) = -\frac{p(\hat{s}(t))}{1-p(\hat{s}(t))}\lambda\hat{s}(t)(1-\hat{s}(t))$ . Panel B plots profits of the boundary initiating bidder (the blue normal line) and a non-initiating bidder (the red dashed line) as a function of the boundary type.  $\hat{s}(\infty)$  is the stationary boundary type.

initial condition  $\hat{s}(0) = 1$ . Its solution gives the evolution of  $\hat{s}(t)$  over time. This equation has an intuitive interpretation. It states that for the indifferent type  $\hat{s}$ , the cost of waiting equals the benefit. The cost of waiting (the left-hand side) consists of three components: delay in the surplus realized from the auction, the possibility that a rival bidder of the high type appears, and the decrease in the continuation value brought about by the change in the initiation threshold. The benefit of waiting is that the bidder delays expecting the seller to initiate the auction, as this leads to a higher expected payoff for the bidder. In addition, the bidder may get a shock and obtain the payoff of  $X$  upon it. This term is positive if the valuation of the bidder is low and negative if the valuation of the bidder is high. Section 5.4 illustrates the solution specializing to the case  $N = 2$  and  $v(s) = s$ . Figure 8 illustrates the behavior of  $p(t)$  as a function of  $\hat{s}(t)$ , as well as the behavior of profits of boundary initiating and non-initiating bidders, for this case, where  $r = 0.05$ ,  $\lambda = 0.5$  (bidders change type on average every two years),  $\mu = 0.2$  (sellers put their asset for sale on average after five years),  $X = 0.15$ .

The following proposition presents some analysis of (49):

**Proposition 6.** *Let  $\hat{s}(t)$  be the equilibrium time-dependent cut-off type in the non-stationary setting. It has the following properties:*

1.  $\hat{s}(t) > 0$ ;
2.  $\hat{s}'(t) = 0$  if the stationary cut-off type,  $\hat{s}$ , is equal to 1.  $\hat{s}'(t) < 0$  if the stationary cut-off type,  $\hat{s}$ , is less than 1.
3. As  $t \rightarrow \infty$ ,  $\hat{s}(t) \rightarrow \hat{s}$ .
4. If the equilibrium cut-off type  $\hat{s}(t)$  for a given  $\mu$ , denoted  $\hat{s}(t, \mu)$ , is unique, then  $\hat{s}(t, \mu)$  is always increasing in  $\mu$  and strictly increasing if  $\hat{s}(t, \mu) < 1$ .

**Proof.** [TO BE COMPLETED]

#### 4.3.4 The seller's initiation problem

[TO BE COMPLETED]



### 4.3.5 Equilibria

[TO BE COMPLETED]

## 5 Special case: Two bidders and linear valuations

In the special case of  $N = 2$  and  $v(s) = s$ , closed-form solutions are attainable. Thus, it is useful to illustrate the results using this special case.

### 5.1 Equilibria in auctions, stationary setting

Consider a bidder-initiated auction. The auction becomes a special case of a problem studied in Kaplan and Zamir (2012). Equations (18) and (22) simplify to

$$\begin{aligned}\gamma_{N,1}(b, \hat{s})(\gamma_I(b, \hat{s}) - b) &= \gamma_N(b, \hat{s}), \\ \gamma_{I,1}(b, \hat{s})(\gamma_N(b, \hat{s}) - b) &= \gamma_I(b, \hat{s}) - \hat{s}.\end{aligned}$$

Adding up the equations and integrating,

$$\gamma_N(b, \hat{s}) \gamma_I(b, \hat{s}) = (\gamma_N(b, \hat{s}) + \gamma_I(b, \hat{s})) b - \hat{s}b + c, \quad (50)$$

where  $c$  is the constant of integration. Boundary condition (27) yields

$$\underline{b}(\hat{s}) = \frac{\hat{s}}{2}.$$

Evaluating (50) at  $b = \underline{b}(\hat{s})$  and using the boundary conditions  $\gamma_I(\underline{b}(\hat{s})) = \hat{s}$  and  $\gamma_N(\underline{b}(\hat{s})) = \underline{b}(\hat{s})$ , we obtain  $c = \frac{\hat{s}^2}{4}$ . Evaluating (50) at  $b = \bar{b}(\hat{s})$  and using the boundary conditions  $\gamma_I(\bar{b}(\hat{s})) = 1$  and  $\gamma_N(\bar{b}(\hat{s})) = \hat{s}$ , we obtain

$$\bar{b}(\hat{s}) = \hat{s} - \frac{\hat{s}^2}{4}.$$

This gives us the range of bids.

From (50):

$$\gamma_N(b) = \frac{b\gamma_I(b) - b\hat{s} + c}{\gamma_I(b) - b}.$$

Plugging into the second differential equation in the original system:

$$\gamma_I'(b) (b^2 - b\hat{s} + c) = (\gamma_I(b) - \hat{s}) (\gamma_I(b) - b).$$

The solution (Kaplan and Zamir, 2012) is

$$\gamma_I(b) = \hat{s} + \frac{\hat{s}^2}{(\hat{s} - 2b) c_I e^{-\frac{\hat{s}}{\hat{s}-2b}} - 4b}.$$

Similarly, the solution for  $\gamma_N(b)$  is

$$\gamma_N(b) = \frac{\hat{s}^2}{(\hat{s} - 2b) c_N e^{\frac{\hat{s}}{\hat{s}-2b}} + 4(\hat{s} - b)},$$

where constants  $c_N$  and  $c_I$  are determined from the initial value conditions  $\gamma_I\left(\hat{s} - \frac{\hat{s}^2}{4}\right) = 1$  and  $\gamma_N\left(\hat{s} - \frac{\hat{s}^2}{4}\right) = \hat{s}$ .

If the auction is seller-initiated, in equilibrium each bidder bids half of her valuation (e.g., Krishna, 2010)

$$\beta(s) = \frac{1}{2}s.$$

It is interesting to compare the payoffs of the boundary type  $\hat{s}$  in all three cases: if she approached the seller, if another bidder approached the seller, and if the seller put the asset for sale without bidder approached by anyone.

$$\begin{aligned} \Pi_I(\hat{s}, \hat{s}) &= \frac{\underline{b}(\hat{s})}{\hat{s}} (\hat{s} - \underline{b}(\hat{s})) = \frac{\hat{s}}{4}, \\ \Pi_N(\hat{s}, \hat{s}) &= \hat{s} - \bar{b}(\hat{s}) = \frac{\hat{s}^2}{4}, \\ \Pi(\hat{s}, \hat{s}) &= \hat{s} - \beta(\hat{s}) = \frac{\hat{s}}{2}. \end{aligned}$$

It is easy to see that  $\Pi(\hat{s}, \hat{s}) > \Pi_I(\hat{s}, \hat{s}) > \Pi_N(\hat{s}, \hat{s})$ , as shown in Lemmas 4 and 6 for general functions and the number of bidders.

## 5.2 The initiation game, stationary setting

As before, consider first the decision problem for a bidder for a given  $\mu$ . For the special case of  $N = 2$  and  $v(s) = s$ , we can plug in the closed-form values of  $\Pi_I(\hat{s}, \hat{s})$ ,  $\Pi_N(\hat{s}, \hat{s})$ , and  $\Pi(\hat{s}, \hat{s})$  into equation (31). It simplifies to

$$r \frac{\hat{s}}{4} + \lambda (1 - \hat{s}) \left( \frac{\hat{s}}{4} - \frac{\hat{s}^2}{4} \right) = \mu \left( \frac{\hat{s}}{2} - \frac{\hat{s}}{4} \right) + \lambda \left( X - \frac{\hat{s}}{4} \right).$$

Simplifying,

$$(1 - \hat{s})^2 \frac{\hat{s}}{4} = X - \frac{r + \lambda - \mu}{\lambda} \frac{\hat{s}}{4} \quad (51)$$

The left-hand side of (51) is independent of the parameters. On  $\hat{s} \in [0, 1]$ , it is an inverted U-shaped function of  $\hat{s}$  that reaches its maximum,  $\frac{1}{27}$ , at  $\hat{s} = \frac{1}{3}$ . In addition, if the right-hand side of (51) exceeds zero at  $\hat{s} = 1$ , there is a potential boundary equilibrium.

The intuition behind the potential multiplicity of equilibria lies in strategic complementarity of initiation decisions among bidders. Note that  $\Pi_I(\hat{s}, \hat{s}) - \Pi_N(\hat{s}, \hat{s})$  is inverted U-shaped on  $\hat{s} \in [0, 1]$ . If a bidder expects other bidders to not approach the seller unless  $s$  is extremely high (i.e.,  $\hat{s} \rightarrow 1$ ), then the payoff of the cut-off type when she initiated the auction versus when she waits until another bidder initiates the auction are close. Thus, a bidder also has lower incentives to approach the seller. In contrast, if  $\hat{s}$  is in the middle so that  $\Pi_I(\hat{s}, \hat{s}) - \Pi_N(\hat{s}, \hat{s})$  is high, then a bidder also has incentives to approach the seller, because waiting results in the significant risk of a reduction in the payoff, if another bidder approaches the seller. Note that this argument is the opposite of that in the common-values model. There initiation decisions of bidders were substitutes in the sense that if a bidder expected another bidder to approach the seller, she had lower incentives to approach the seller herself.

Because the expected revenues of the seller are not computable in closed-form, the characterization of the seller's problem is not simplified compared to the general case.

## 5.3 Equilibria in auctions, off-steady state setting

As before,  $\underline{b} = \frac{\hat{s}}{2}$ . Multiplying (37) by  $\frac{1-\hat{s}}{1-p}$ , multiplying (39) by  $\hat{s}$  and adding the equations up, we obtain (slightly abusing notation and omitting the last two arguments in  $F_N$ ,  $F_I$ ,  $\underline{b}$ ,

$\bar{b}$ , and  $\hat{b}$ ):

$$\begin{aligned}
\frac{1-\hat{s}}{1-p} F_I'(b) (F_N(b) \hat{s} - b) + \hat{s} F_N'(b) \left( \hat{s} + \frac{F_I(b) - p}{1-p} (1 - \hat{s}) - b \right) &= \frac{1-\hat{s}}{1-p} F_I(b) + \hat{s} F_N(b) \\
d \left( \hat{s} F_N(b) \left( \hat{s} + \frac{F_I(b) - p}{1-p} (1 - \hat{s}) \right) \right) &= d \left( \frac{1-\hat{s}}{1-p} F_I(b) b + \hat{s} F_N(b) b \right) \\
\hat{s} F_N(b) \left( \hat{s} + \frac{F_I(b) - p}{1-p} (1 - \hat{s}) \right) &= \frac{1-\hat{s}}{1-p} F_I(b) b + \hat{s} F_N(b) b \quad (52)
\end{aligned}$$

for some constant  $D$ . This equation holds for any  $b \in [\hat{b}, \bar{b}]$ . Evaluating at  $\bar{b}$ ,

$$\begin{aligned}
\hat{s} &= \frac{1-\hat{s}}{1-p} \bar{b} + \hat{s} \bar{b} + D \\
D &= \hat{s} (1 - \bar{b}) - \frac{1-\hat{s}}{1-p} \bar{b}.
\end{aligned}$$

Evaluating at  $\hat{b}$ ,

$$\begin{aligned}
\hat{s} (\hat{s} - \hat{b}) \frac{b(\hat{s} - \bar{b})}{\hat{s}(\hat{s} - \hat{b})} &= \frac{1-\hat{s}}{1-p} p \hat{b} + D \\
D &= \frac{\hat{s}^2}{4} - \frac{1-\hat{s}}{1-p} p \hat{b} \Rightarrow \\
\bar{b} &= \frac{\hat{s} - \frac{\hat{s}^2}{4} + \frac{1-\hat{s}}{1-p} p \hat{b}}{\hat{s} + \frac{1-\hat{s}}{1-p}}. \quad (53)
\end{aligned}$$

This equation pins down  $\bar{b}$ , given  $\hat{b}$ . In the stationary setting,  $p = 0$  so  $\bar{b} = \frac{\hat{s}^2}{4} - \hat{s}$ . We are left with determining the remaining value,  $\hat{b}$ . Going back to equation (52),

$$\hat{s} F_N(b) \left( \hat{s} - b + \frac{F_I(b) - p}{1-p} (1 - \hat{s}) \right) = \frac{1-\hat{s}}{1-p} F_I(b) b + \frac{\hat{s}^2}{4} - \frac{1-\hat{s}}{1-p} p \hat{b},$$

implying that in the range  $b \in [\hat{b}, \bar{b}]$ :

$$\hat{s} F_N(b) = \frac{\frac{1-\hat{s}}{1-p} F_I(b) b + \frac{\hat{s}^2}{4} - \frac{1-\hat{s}}{1-p} p \hat{b}}{\hat{s} - b + \frac{F_I(b) - p}{1-p} (1 - \hat{s})}$$

$$\begin{aligned}
F'_I(b) \left( \frac{\frac{1-\hat{s}}{1-p} F_I(b) b + \frac{\hat{s}^2}{4} - \frac{1-\hat{s}}{1-p} p \hat{b}}{\hat{s} - b + \frac{F_I(b)-p}{1-p} (1-\hat{s})} - b \right) &= F_I(b) \\
F'_I(b) \frac{\frac{1-\hat{s}}{1-p} F_I(b) b + \frac{\hat{s}^2}{4} - \frac{1-\hat{s}}{1-p} p \hat{b} - b \hat{s} + b^2 - \frac{F_I(b)-p}{1-p} (1-\hat{s}) b}{\hat{s} - b + \frac{F_I(b)-p}{1-p} (1-\hat{s})} &= F_I(b) \\
F'_I(b) \frac{\frac{\hat{s}^2}{4} - \frac{1-\hat{s}}{1-p} p \hat{b} - b \hat{s} + b^2 + \frac{p}{1-p} (1-\hat{s}) b}{\hat{s} - b + \frac{F_I(b)-p}{1-p} (1-\hat{s})} &= F_I(b), \\
F'_I(b) &= \frac{\left( \hat{s} - b + \frac{F_I(b)-p}{1-p} (1-\hat{s}) \right) F_I(b)}{\frac{\hat{s}^2}{4} - \frac{1-\hat{s}}{1-p} p \hat{b} - b \hat{s} + b^2 + \frac{p}{1-p} (1-\hat{s}) b}.
\end{aligned}$$

The boundary condition is

$$F_I(\bar{b}(\hat{b})) = F_I \left( \frac{\hat{s} - \frac{\hat{s}^2}{4} + \frac{1-\hat{s}}{1-p} p \hat{b}}{\hat{s} + \frac{1-\hat{s}}{1-p}} \right) = 1.$$

The above differential equation has a closed-form solution which is omitted for brevity and available from the authors upon request. The remaining value,  $\hat{b}$ , is numerically determined from the second boundary condition:

$$F_I(\hat{b}) = p.$$

It can be shown that  $\frac{\hat{s}}{2} \leq \hat{b} \leq \frac{3\hat{s}}{4}$  and  $\hat{b}(p)$  is strictly increasing, with the first (second) condition holding as equality for  $p = 0$  ( $p = 1$ );  $\frac{3\hat{s}}{4} \leq \bar{b}$  and  $\bar{b}(p)$  is strictly decreasing, with the condition holding as equality if  $p = 1$ .

If the auction is seller-initiated, as before,

$$\beta(s) = \frac{1}{2}s.$$

The payoffs of the boundary type  $\hat{s}$  in all three cases: if she approached the seller, if another bidder approached the seller, and if the seller put the asset for sale without bidder

approached by anyone, are as follows:

$$\begin{aligned}\Pi_I(\hat{s}, \hat{s}, p) &= \frac{\underline{b}(\hat{s}, p)}{\hat{s}} (\hat{s} - \underline{b}(\hat{s}, p)) = \frac{\hat{s}}{4}, \\ \Pi_N(\hat{s}, \hat{s}, p) &= \hat{s} - \bar{b}(\hat{s}, p) < \Pi_I(\hat{s}, \hat{s}, p), \\ \Pi(\hat{s}, \hat{s}) &= \hat{s} - \beta(\hat{s}) = \frac{\hat{s}}{2} > \Pi_I(\hat{s}, \hat{s}, p).\end{aligned}$$

## 5.4 The initiation game, off-steady state setting

For a given  $\mu$ ,  $N = 2$ , and  $v(s) = s$ , equation (49) simplifies to

$$r \frac{\hat{s}}{4} + \lambda \frac{1 - \hat{s}}{1 - p} \left( \bar{b}(\hat{s}, p) - \frac{3\hat{s}}{4} \right) - \frac{p}{1 - p} \lambda \hat{s}(1 - \hat{s}) \frac{1}{2} \left( \frac{\hat{b}(\hat{s}, p) - \hat{s}/2}{\hat{s} - \hat{b}(\hat{s}, p)} + \frac{1}{2} \right) = \mu \left( \frac{\hat{s}}{2} - \frac{\hat{s}}{4} \right) + \lambda \left( X - \frac{\hat{s}}{4} \right).$$

This equation sets up a differential equation for  $\hat{s}(t)$ . Specifically, from the above equation,  $p(\hat{s})$  can be obtained for any value of  $\hat{s}$  and then, a differential equation  $\hat{s}'(t) = -\frac{p(\hat{s}(t))}{1 - p(\hat{s}(t))} \lambda \hat{s}(t)(1 - \hat{s}(t))$  can be numerically solved with the initial condition  $\hat{s}(0) = 1$ .

# 6 Model Implications

In this section, we discuss implications of the model.

## 6.1 Efficiency of Takeovers and the Role for Shareholder Activism

The model implies that bidders are strongly disincentivized to approach targets in common-value environments. What are the situations that represent these environments? Motivations for acquisitions can be broadly divided into two groups. First, acquisitions can be motivated by synergies of a target with a particular bidder, such as potential gains from combining technologies. Because high synergies with one bidder do not necessarily mean high synergies with the other bidder, acquisitions driven by synergy motivations are closer to the private-value setting. Second, acquisitions can be motivated by poor performance of the incumbent management of the target. Because gains from replacing bad management and/or changing policies are likely to be similar for different bidders, who, however, may have different estimates of the value created, such environments are closer to the common-value setting. Thus, the paper implies that it is precisely the second type of targets that

potential bidders have little incentives to initiate.

Assuming for a moment that inefficiently managed targets are represented well by the pure common-values setting, the only way for such deals to take place is to be initiated by the target. If, however, the management is entrenched and wants to preserve independence, the target has little incentive to initiate the deal. As a consequence, inefficiently-run companies can remain without a change in ownership for a long time, even if gains from the change in ownership are substantial. Thus, the role of takeovers as a corporate governance mechanism can be limited. In contrast to the free-rider problem in tender offers that applies uniformly to both common- and private-value takeover bids in the form of tender offers, the problem that we emphasize is centered around common-value takeovers; the effect is limited in private-value settings, such as when synergies are bidder-specific.

The lack of incentive for bidders to initiate common-value auctions gives rise to alternative ways of promoting takeovers. In particular, it gives rise to shareholder activism. Consider the target with an extremely entrenched management, such that it never wants to put itself for sale no matter how high the potential surplus from the sale is. However, the target can be forced to put itself for sale if the board is pressured by a blockholder, such as an activist hedge fund. In this context, one thinks about cost  $I$ , introduced in Corollary to Proposition 1, as the cost that a shareholder (or a potential shareholder) needs to incur to convince the board of the target to sell itself. A case in point is the acquisition of Blue Coat by Thoma Bravo, discussed in the introduction. In that case, an activist hedge fund Elliot Associates accumulated a 9% ownership stake in Blue Coat and forced the board of Blue Coat to auction the company off.

While the model focuses on a single target, it is straightforward to apply its results for the problem of selecting one target out of a multitude of potential targets by a bidder. Because of unraveling of initiation in common-value auctions, the model implies that bidders will tend to approach targets, in which they have a substantive private value component of valuation, even if other potential targets have considerably greater potential gains from the acquisition.

## 6.2 Empirical Predictions about Initiation

Because the degree of the common-value component in the valuation is not directly observable, it needs to be proxied in empirical tests. A good proxy is whether the takeover

battle is between strategic bidders or financial (private equity) bidders. Intuitively, because different private equity firms tend to use similar strategies after they acquire the target, their valuations should have a significant common component, even though they may have different estimates of it. Given this, the model leads to the following predictions:

1. *Contests among financial bidders are more likely to be target-initiated than contests among strategic bidders.*
2. *In bidder-initiated PE deals, the initiating bidder is very likely to have a toehold.*
3. *A strategic bidder approaches targets in which she has a high private component of the valuation (even if synergies with these are lower than with alternatives).*

The first prediction is a direct consequence of the model. It is consistent with the summary statistics on parties initiating takeovers in Fidrmuc et al. (2012). The second prediction holds because a toehold allows the initiating bidder to preserve part of the surplus, even if his type is the lowest.<sup>11</sup> The last prediction follows from the discussion in Section 6.1.

### 6.3 Empirical Predictions about Bidding

The model leads to a number of implications about how bidding is different in bidder- versus seller-initiated takeover auctions. They are driven by endogenous initiation of takeovers by bidders and the resulting asymmetries between the initiating bidder and his competitors.

1. *All else equal, bidders in seller-initiated auctions are weaker (have lower valuations) than bidders in bidder-initiated auctions.*
2. *Conditional on the same valuations, bidders bid less aggressively in seller-initiated deals.*
3. *In bidder-initiated deals, the initiating bidder is stronger (has, on average, higher valuations) than the other bidders.*

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<sup>11</sup>An earlier draft of this paper contains the formal model of toeholds in a simplified auction environment. The results are available upon request.



4. *In bidder-initiated deals, conditional on the same valuation, the non-initiating bidder bids more aggressively than the initiating bidder:  $\beta_N(\hat{s}, \hat{s}) > \beta_I(\hat{s}, \hat{s})$ .*
5. *In bidder-initiated deals, unconditionally on the exact valuation, the initiating bidder bid more aggressively and wins more often:  $\mathbb{E}[\beta_I(s, \hat{s})] > \mathbb{E}[\beta_N(s, \hat{s})]$ .*

## 7 Conclusion

In this paper, we examine theoretically endogenous initiation of a first-price auction by potential buyers and the seller. Each buyer has an option to approach the seller, sending a message that will trigger the auction. Alternatively, the seller can choose to put the asset for sale without waiting to be approached by a potential buyer. Our framework aims to capture many real-world environments in which initiation of an auction is a strategic choice. Examples include corporate takeover and intercorporate asset sales, as well as auctions of art. Our main results relate information effects of bidder initiation or lack of thereof to the valuation framework. We show that in a “common-values” auction, such as a battle between several financial bidders for the target company, bidders are reluctant to approach the seller, because it erodes their information rents. This effect is extreme: unraveling occurs and no bidder ever approaches the seller. All auctions are initiated by the seller, if at all. In particular, the timing of sale is uninformative in the sense that it is not sensitive to bidders’ valuations. By contrast, in a “private-values” auction, the effect can be opposite. Observing that no bidder has approached the seller yet reveals information that competitors are weak, which incentivizes a bidder with a high enough valuation to approach the seller. We also show that bidder- and seller-initiation are substitutes, so several equilibria, which differ in how the auctions occur, can coexist in the “private-values” framework. Finally, we derive a number of implications relating the initiating party to bids and auction outcomes.

## Appendix

**Example of the common-value setting:**  $v(s_1, s_2) = \frac{1}{2}(s_1 + s_2)$ . To find the equilibrium, let  $\phi_i(b, \hat{s}) = \alpha_i + \beta_i b$ . Plugging in to the boundary conditions and differential

equations yields The first boundary condition is:

$$\begin{aligned}\hat{s} &= \alpha_1 + \beta_1 \frac{1}{2} \hat{s}, \\ 0 &= \alpha_2 + \beta_2 \frac{1}{2} \hat{s}. \\ \hat{s} &= \alpha_1 + \alpha_2 + (\beta_1 + \beta_2) \frac{\hat{s}}{2}\end{aligned}$$

The second boundary condition:

$$\begin{aligned}1 &= \alpha_1 + \beta_1 \bar{b} \\ \hat{s} &= \alpha_2 + \beta_2 \bar{b} \\ 1 + \hat{s} &= \alpha_1 + \alpha_2 + (\beta_1 + \beta_2) \bar{b}\end{aligned}$$

Pluggin into the differential equation yields:

- Bidder 1's strategy:

$$\phi_1(b, \hat{s}) = \hat{s}(2\hat{s} - 1) + 4(1 - \hat{s})b;$$

- Bidder 2's strategy:

$$\phi_2(b, \hat{s}) = -2\hat{s}^2 + 4\hat{s}b.$$

For both strategies,  $b \in [\frac{1}{2}\hat{s}, \frac{1+2\hat{s}}{4}]$ . The bidding strategies given signals are inverses of  $\phi_1(b)$  and  $\phi_2(b)$ :

$$\begin{aligned}\beta_1(s, \hat{s}) &= \frac{s + \hat{s}(1 - 2\hat{s})}{4(1 - \hat{s})}; \\ \beta_2(s, \hat{s}) &= \frac{s + 2\hat{s}^2}{4\hat{s}}.\end{aligned}$$

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